

Second short test 2022

Exercise 1

A: heads at most once

B: both heads and tails at least once

A and B are independent if

$$P(A \cap B) = P(A)P(B)$$

$$\begin{aligned} P(A) &= P(3 \text{ times tails}) + P(1 \text{ heads, 2 tails}) \\ &= \left(\frac{1}{2}\right)^3 + \binom{3}{1} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{8} + \frac{3}{8} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} P(B) &= P(2 \text{ times heads, one time tails}) + P(2 \text{ tails, 1 heads}) \\ &= 3 \cdot \left(\frac{1}{2}\right)^3 + 3 \left(\frac{1}{2}\right)^3 = \frac{3}{4} \end{aligned}$$

$$(\text{or } 1 - P(\text{all heads}) - P(\text{all tails}) = 1 - \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^3)$$

$$P(A \cap B) = P(1 \text{ heads, 2 tails}) = 3 \left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$P(A \cap B) = \frac{3}{8} \quad P(A)P(B) = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}, \text{ So indeed independent.}$$

Ex 2

Let X be the number of blue balls taken from the first urn

$$P(X=0) = \frac{\binom{3}{3}\binom{2}{0}}{\binom{5}{3}} = \frac{1}{10}$$

$$P(X=1) = \frac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}} = \frac{6}{10}$$

$$P(X=2) = \frac{\binom{3}{1}\binom{2}{2}}{\binom{5}{3}} = \frac{3}{10}$$

$$P(X=1|Y=2) = \frac{P(X=1 \cap Y=2)}{P(Y=2)}$$

$$= \frac{P(X=1)P(Y=2|X=1)}{P(X=0)P(Y=2|X=0) + P(X=1)P(Y=2|X=1) + P(X=2)P(Y=2|X=2)}$$

$$= \frac{\frac{6}{10} \binom{3}{1} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)}{\frac{1}{10} \cdot 0 + \frac{6}{10} \binom{3}{1} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right) + \frac{3}{10} \binom{3}{1} \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3}}$$

$$\left(= \frac{\frac{3}{5} \cdot \frac{2}{9}}{\frac{3}{5} \cdot \frac{2}{9} + \frac{3}{10} \cdot \frac{4}{9}} = \frac{\frac{6}{45}}{\frac{6}{45} + \frac{6}{45}} = \frac{1}{2} \right)$$