First short test 2022

Exercise 1

(a) $P(A) = 1 - P(A^c)$

$$= 1 - \frac{\#(A^c)}{\#(\Omega)} = 1 - \frac{5^5}{6^5}$$

$(\Omega = \{(x_1, \dots, x_5) : x_1, \dots, x_5 \in \{1, \dots, 6\}\})$

or by multiplying probabilities
$1 - \left(\frac{5}{6}\right)^5.$

or $\frac{\binom{5}{1} \cdot 5^4}{6^5} + \frac{\binom{5}{2} \cdot 5^3}{6^5} + \frac{\binom{5}{3} 5^2}{6^5} + \frac{\binom{5}{4} \cdot 5}{6^5} + \frac{1}{6^5}.$

(b) $P(B) = \frac{\binom{5}{3} \cdot 5^2}{6^5}$ $\left(\text{or } \binom{5}{3} \cdot \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2\right).$

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A)$ and $P(B)$ as above
$P(A \cap B) = P(3 \text{ times } '4' \ 2 \text{ times } '2')$
$+ P(3 \text{ times } '4' \ 1 \ '2' \ 1 \text{ no } '2' \text{ and no } '4')$

$$= \frac{\binom{5}{3}}{6^5} + \frac{\binom{5}{3} \cdot 1 \cdot \overset{\text{no 2 or 4}}{4} \cdot \overset{\text{2 possible positions of the '2' after you placed the '4's}}{2}}{6^5}$$

Exercise 2

$$P(W_1 \mid X=2) = \frac{P(W_1 \cap X=2)}{P(X=2)}$$

All $7\cdot6\cdot5$ possible outcomes have equal probability

Write W_i for the event that the i^{th} ball is white
R_i for the event that the i^{th} ball is red

$$\frac{P(W_1 \cap X=2)}{P(X=2)} = \frac{P(W_1W_2R_3)+P(W_1R_2W_3)}{P(W_1W_2R_3)+P(R_1W_2W_3)+P(W_1R_2W_3)}$$

$$= \frac{\frac{3\cdot2\cdot4}{7\cdot6\cdot5}+\frac{3\cdot4\cdot2}{7\cdot6\cdot5}}{\frac{3\cdot2\cdot4}{7\cdot6\cdot5}+\frac{4\cdot3\cdot2}{7\cdot6\cdot5}+\frac{3\cdot4\cdot2}{7\cdot6\cdot5}} = \frac{2}{3}.$$

You can also take $\frac{\#(W_1 \cap X=2)}{\#(X=2)}$ leaving out the $7\cdot6\cdot5$ everywhere.

Or you can use cond probabilities instead of counting e.g. $P(W_1W_2R_3) = P(W_1)P(W_2 \mid W_1)P(R_3 \mid W_1W_2)$
$$= \frac{3}{7}\cdot\frac{2}{6}\cdot\frac{4}{5}.$$