Final Exam Probability Theory

July 1, 2021, 8.30-11.15 (extra time 8.30-11.45)

- This exam consists of seven questions and a table. You can score 45 points. Your grade is given by (5+number of points)/5.
- You may use a simple calculator, but a graphical or programmable calculator is not allowed.
- Explain your answers clearly.
- You have to keep track of the time yourself. Make sure that you stop at 11.15 (or 11.45 if you have extra time) and upload your answers as one single pdf via Canvas within 10 minutes.
- If you want to stop before the end time of the exam, ask permission of your TA via the chat. After the TA gives you permission, upload your answers as one single pdf via canvas within 10 minutes and additionally let the TA know in the chat when you finished uploading. You can then leave the meeting, but it is not allowed to have any communication with other students about the exam before 12.30.
- If you need to go to the bathroom, let the TA know in the chat and wait for permission. If you don't get permission within 2 minutes, just go and mention when you are back.
- It is of great help if you start each of the seven exercises on a new page!
- Good luck!

1. We have an urn with 3 green and 2 yellow balls. We pick a sample of 2 balls *without replacement* and put these balls in a second urn that was previously empty. Next we sample 3 balls from this second urn *with replacement*.
(a) [3 points] Compute the probability that we obtain 3 yellow balls in the second sample.
(b) [4 points] Compute the conditional probability that the balls obtained in the first sample have the same colour given that the three balls obtained from the second urn have the same colour.

2. An interviewer is given a list of 6 people she can interview. She needs to interview 4 people in total and will ask the people on her list one by one whether they agree to be interviewed. She stops as soon as 4 people have agreed to be interviewed. Suppose that each person (independently) agrees to be interviewed with probability $\frac{2}{3}$.
(a) [4 points] Compute the probability that her list will enable her to obtain the necessary number of interviews.
(b) [3 points] Let A be the event that the first person she asks agrees to be interviewed and B the event that she will contact exactly 5 people. Compute $P(A \cup B)$.

3. If you bet €1 on red in the game of roulette, this means that you win €1 if red appears and lose €1 (so 'win' €−1) if red does not appear. A gambler recommends the following 'winning strategy': Bet €1 on red. If red appears (which has probability $\frac{18}{38}$), then take the €1 profit and quit. When red does not appear and you lose this bet (which has probability $\frac{20}{38}$ of occuring), you make an additional €1 bet on red on **each** of the next **two** spins of the roulette wheel and then quit. Let X be your winnings if you quit.
(a) [3 points] Compute $P(X > 0)$.
(b) [3 points] Compute $E(X)$.

4. Let X be exponentially distributed with parameter 4 and let $Y := e^X - 1$.
(a) [3 points] Compute the density function of Y.
(b) [3 points] Compute the variance of $2Y$.

5. [4 points] On each bet, a gambler loses €1 with probability $\frac{7}{10}$, loses €2 with probability $\frac{2}{10}$ and wins €10 with probability $\frac{1}{10}$. Use the Central Limit Theorem and the table to approximate the probability that, after 100 games, he lost more euro than he won. You don't have to use the continuity correction.

6. Let the random variables X and Y have joint density function

$$f_{X,Y}(x, y) = \begin{cases} (y+1)e^{-yx-x-y} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Show that Y is exponentially distributed with parameter 1.
(b) [3 points] Find $E(X|Y = y)$ for $y \geq 0$.

7. Let X and Y be independent random variables, where X is uniformly distributed on $[0, 1]$ and the density function of Y is given by

$$f_Y(y) = \begin{cases} 2y & \text{if } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Let f_Z be the density function of $Z := X + Y$. Compute $f_Z(z)$ for $1 \leq z \leq 2$.
(b) [3 points] Compute $\text{Cov}(XY, X)$.
(c) [3 points] Compute $P(Y - X < 0)$.

Appendix E

Table of values for $\Phi(x)$

$\Phi(x) = P(Z \leq x)$ is the cumulative distribution function of the standard normal random variable Z.

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998