Final Probability theory July 1.

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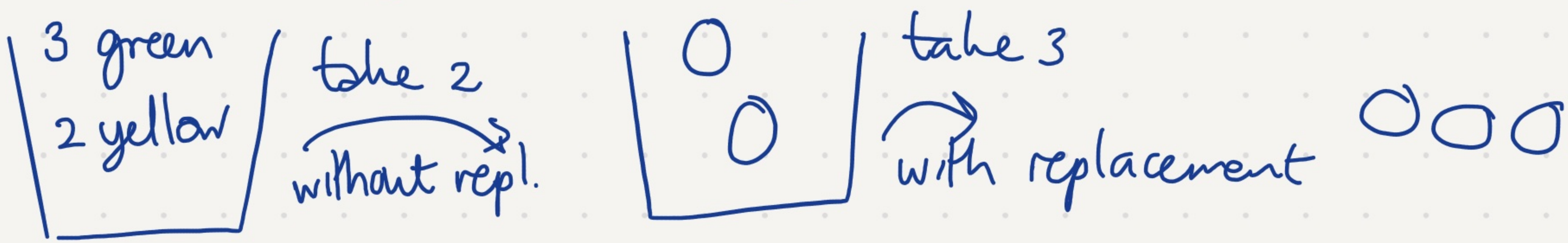

(a) P(three yellow second sample) =

P(first sample two yellow)P(2^{nd} sample 3 yellow | 1^{st} sample 2 yellow)

+ P(first sample one green one yellow) P(2^{nd} sample 3 yellow | 1^{st} sample one gr one yellow)

$$= \frac{\binom{2}{2}}{\binom{5}{2}} \cdot 1 + \frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} \cdot \left(\frac{1}{2}\right)^3$$

$$= \frac{1}{10} + \frac{6}{10} \cdot \frac{1}{8} = \frac{1}{10} + \frac{3}{40} = \frac{7}{40}.$$

b) A: all balls in first sample have the same colour
B: all balls in second sample have the same colour

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$P(A \cap B) = P(A) = P(\text{first sample 2 green}) + P(\text{first sample 2 yellow})$

$$= \frac{\binom{3}{2}}{\binom{5}{2}} + \frac{\binom{2}{2}}{\binom{5}{2}} = \frac{3}{10} + \frac{1}{10} = \frac{2}{5}.$$

$P(B) = P(A) + P(\text{first sample 2 diff colors, second all same})$

$= P(A) + P(\text{first sample yellow+green, second all green})$
$+ P(\text{first sample yellow + green, second all yellow})$

$$= \frac{2}{5} + \frac{3}{5} \cdot \frac{1}{8} + \frac{3}{5} \cdot \frac{1}{8} = \frac{2}{5} + \frac{6}{40} = \frac{2}{5} + \frac{3}{20} = \frac{11}{20}.$$

or $P(B) = P(A)\,P(B|A) + P(A^C)\,P(B|A^C)$

$$= \frac{2}{5} \cdot 1 + \frac{3}{5} \cdot \frac{1}{4}$$

first ball is fine, second and third need to be same as first.

So $P(B|A) = \dfrac{\frac{2}{5}}{\frac{11}{20}} = \dfrac{2}{5} \cdot \dfrac{20}{11} = \dfrac{8}{11}$.

2 a) P(enough people on list)

= P(first 4 agree) +

P (exactly 3 of first 4 agree , fifth agrees)

+ P (exactly 3 of first 5 agree , sixth agrees)

$$= \left(\frac{2}{3}\right)^4 + \binom{4}{3} \cdot \left(\frac{2}{3}\right)^3 \cdot \frac{1}{3} \cdot \frac{2}{3}{}^{1} + \binom{5}{3} \cdot \left(\frac{2}{3}\right)^3 \cdot \frac{1}{3} \cdot \frac{2}{3}$$

$$= \frac{16}{81} + 4 \cdot \frac{16}{243} + 10 \cdot \frac{16}{729} = \frac{16 \cdot 9 + 4 \cdot 16 \cdot 3 + 10 \cdot 16}{729}$$

$$= \frac{496}{729} \approx 0{,}68$$

b) A: first person agrees

B: she contacts exactly 5 people

$$P(A \cup B) = P(A) + P(A^C \cap B)$$

$$= \frac{2}{3} + P(\text{first does not agree, 2, 3, 4, 5 agree})$$

$$= \frac{2}{3} + \frac{1}{3}\left(\frac{2}{3}\right)^4 = \frac{2 \cdot 3^4 + 16}{3^5} = \frac{178}{243}$$

$$\approx 0{,}73.$$

Or $P(A \cup B) = P(A) + P(B) + P(A \cap B)$

$P(A) = \frac{2}{3}$ $\quad P(B) = \binom{4}{3} \cdot \left(\frac{2}{3}\right)^3 \cdot \frac{1}{3} \cdot \frac{2}{3} = 4 \cdot \frac{16}{243}$

$P(A \cap B) = P(\text{first agrees, 2 of 2,3,4 agree, 5th agrees})$

$$= \frac{2}{3} \cdot 3 \cdot \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} \cdot \frac{2}{3}.$$

$$\frac{2}{3} + \frac{64}{243} - \frac{48}{243} = \frac{178}{243} \approx 0,73.$$

3 a) $P(X>0) = P(w) + P(lww)$

$$= \frac{18}{38} + \frac{20}{38} \cdot \left(\frac{18}{38}\right)^2$$

$$= \frac{4059}{6859} \approx 0{,}59.$$

b) $P(X=1) = P(X>0) = \frac{4059}{6859}$

$P(X=-3) = P(lll) = \left(\frac{20}{38}\right)^3 = \frac{1000}{6859}$

$P(X=-1) = P(lwl) + P(llw) = 2 \cdot \left(\frac{20}{38}\right)^2 \cdot \left(\frac{18}{38}\right) = \frac{1800}{6859}$

(or $1 - P(X=1) - P(X=-3)$)

$$E(X) = 1 \cdot \frac{4059}{6859} - 3 \cdot \frac{1000}{6859} - 1 \cdot \frac{1800}{6859} = \frac{-741}{6859} \approx -0{,}11.$$

④ $Y = e^X - 1$

For $y \geq 0$: $F_Y(y) = P(Y \leq y) = P(e^X - 1 \leq y) = P(e^X \leq y+1)$

$= P(X \leq \ln(y+1)) = F_X(\ln(y+1))$

for $y < 0$: $f_Y(y) = 0$

for $y \geq -1$: $f_Y(y) = f_X(\ln(y+1)) \cdot \frac{1}{y+1}$

$$= 4e^{-4(\ln(y+1))} \cdot \frac{1}{y+1} = 4 \cdot \frac{(y+1)^{-4}}{y+1}$$

$$= \frac{4}{(y+1)^5}.$$

④ $Var(2Y) = 4\,Var(Y)$

$$Var(Y) = Var(e^X - 1) = Var(e^X).$$
$$= E(e^{2X}) - (E(e^X))^2.$$

$$E(e^{2X}) = \int_0^\infty e^{2X} \cdot 4e^{-4X}\,dx = \int_0^\infty 4e^{-2x}\,dx$$

$$= \left[-2e^{-2X}\right]_0^\infty = 2.$$

$$E(e^X) = \int_0^\infty e^x \cdot 4e^{-4x}\,dx = \int_0^\infty 4e^{-3x}\,dx = \left[-\frac{4}{3}e^{-3X}\right]_0^\infty = \frac{4}{3}.$$

$$Var(Y) = 4 - \left(\frac{4}{3}\right)^2 = 2\frac{2}{9}$$
$$Var(2Y) = 8\frac{8}{9}.$$

⑤ X_i: winnings in i^{th} game

$$E(X_i) = -1 \cdot \frac{7}{10} - 2 \cdot \frac{2}{10} + 10 \cdot \frac{1}{10} = -\frac{1}{10}$$

$$E(X_i^2) = 1 \cdot \frac{7}{10} + 4 \cdot \frac{2}{10} + 100 \cdot \frac{1}{10} = \frac{115}{10} = 11{,}5$$

$$Var(X_i) = \frac{115}{10} - \left(\frac{1}{10}\right)^2 = 11{,}49$$

$$P(X_1 + \cdots + X_{100} < 0) = P\left(\frac{X_1 + \cdots + X_{100} - 100 \cdot \left(\frac{1}{10}\right)}{\sqrt{100} \cdot \sqrt{11{,}49}} < \frac{0 + 100 \cdot \frac{1}{10}}{\sqrt{1149}}\right)$$

$$\approx \Phi(0{,}30) = 0{,}6179$$

(6) (a) For $y \geq 0$

$$f_Y(y) = \int_0^\infty (y+1) e^{-(y+1)x} \cdot e^{-y} \, dx$$

$$= \left[-e^{-(y+1)x} \cdot e^{-y} \right]_{x=0}^{\infty} = e^{-y}$$

So Y is exp with par 1

b) $$f_{X|Y=y}(x|y) = \frac{(y+1)e^{-(y+1)x} \cdot e^{-y}}{e^{-y}} = (y+1)e^{-(y+1)x}$$

for $x \geq 0$

So conditionally on $Y=y$, X is exp with $\lambda = y+1$.

So $E(X|Y=y) = \frac{1}{y+1}$

Or integration by parts

$$E(X|Y=y) = \int_0^\infty x \cdot f_{X|Y}(x|y)\,dx$$

$$= \int_0^\infty x \cdot \frac{(y+1)e^{-(y+1)x} \cdot e^{-y}}{e^{-y}}\,dx = \int_0^1 x \cdot (y+1)e^{-(y+1)x}\,dx$$

$$= \left[x \cdot -e^{-(y+1)x}\right]_0^\infty - \int_0^\infty -e^{-(y+1)x}\,dx$$

$$= \left[-\frac{1}{y+1}e^{-(y+1)x}\right]_0^\infty = \frac{1}{y+1}.$$

7 a $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$

$= \int_0^1 f_Y(z-x)\, dx.$

$z - x \geq 0$ gives
$-x \geq -z$
$x \leq z$. since $z \in [1,2]$ this okay.

$z - x \leq 1$ gives
$-x \leq 1 - z$
$x \geq z - 1.$

$= \int_{z-1}^{1} 2(z-x)\, dx$

$= \left[2zx - x^2\right]_{x=z-1}^{1} = 2z - 1 - (2z(z-1) - (z-1)^2)$

$= 2z - 1 - (2z^2 - 2z - z^2 + 2z - 1)$

$= 2z - z^2.$

Or $f_Z(z) = \int_0^1 f_X(z-y) \cdot 2y \, dy$

$$= \int_{z-1}^{1} 2y \, dy = \left[y^2 \right]_{z-1}^{1} = 1 - (z^2 - 2z + 1) = -z^2 + 2z .$$

b) $\mathrm{Cov}(XY, X) = E(X^2Y) - E(XY) \cdot E(X)$

$E(X) = \frac{1}{2}$

$E(X^2) = \int_0^1 x^2 \cdot 1 \, dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

$E(Y) = \int_0^1 y \cdot 2y \, dy = \left[\frac{2}{3}y^3\right]_0^1 = \frac{2}{3}.$

$E(X^2Y) = E(X^2) \cdot E(Y) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$

$E(XY) = E(X) \cdot E(Y) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

So $\mathrm{Cov}(X,Y) = \frac{2}{9} - \frac{1}{3} \cdot \frac{1}{2} = \frac{2}{9} - \frac{1}{6} = \frac{1}{18}.$

(c) $P(Y - X < 0) = P(Y < X)$

$$= \int_0^1 \int_0^x f_{X,Y}(x,y)\,dy\,dx$$

$$\overset{ind}{=} \int_0^1 \int_0^x f_X(x) f_Y(y)\,dy\,dx$$

$$= \int_0^1 \int_0^x 1 \cdot 2y\,dy\,dx = \int_0^1 [y^2]_0^x\,dx = \int_0^1 x^2\,dx$$

$$= \left[\frac{1}{3} x^3\right]_0^1 = \frac{1}{3}.$$