Final Exam May 27

1 a $\quad f_X(x) = \begin{cases} 3e^{-3x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

$Z := \frac{1}{X+1}$ so Z takes values in $(0, 1]$

For $z \leq 0$: $F_Z(z) = 0$ and for $z \geqslant 1$ $F_Z(z) = 1$.

For $z \in (0, 1)$:

$$F_Z(z) = P(Z \leq z) = P\left(\frac{1}{X+1} \leq z\right) = P\left(X+1 \geqslant \frac{1}{z}\right)$$

$$= P\left(X \geqslant \frac{1}{z} - 1\right) = 1 - F_X\left(\frac{1}{z} - 1\right).$$

So: $f_Z(z) = 0$ for $z < 0$ and $z \geqslant 1$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = -f_X\left(\frac{1}{z} - 1\right) \cdot \frac{d}{dz}\left(\frac{1}{z} - 1\right)$$

$$= -f_X\left(\frac{1}{z} - 1\right) \cdot -\frac{1}{z^2} = \frac{3}{z^2} e^{-3\left(\frac{1}{z} - 1\right)}$$

$$\left(-3\left(\frac{1}{z} - 1\right) = -\frac{3}{z} + 3 \quad \text{or} \quad \frac{-3 + 3z}{z^2} \right).$$

Or via fact 5.24. $g(x) = \frac{1}{x+1}$ gives $g^{-1}(z) = \frac{1}{z} - 1$.
with $z \in (0, 1]$.
also gives $f_Z(z) = 0$ for $z \notin (0, 1)$ and

$$f_Z(z) = f_X\left(\frac{1}{z} - 1\right) \cdot \left| \frac{d}{dz}\left(\frac{1}{z} - 1\right) \right| = \frac{3}{z^2} e^{-3\left(\frac{1}{z} - 1\right)}.$$

b) $\mathrm{Var}(e^{-2X}) = E(e^{-4X}) - (E(e^{-2X}))^2$.

$$E(e^{-4X}) = \int_0^\infty e^{-4x} \cdot 3e^{-3x}\,dx = \int_0^\infty 3e^{-7x}\,dx = \left[-\frac{3}{7} e^{-7x}\right]_0^\infty = \frac{3}{7}.$$

$$E(e^{-2x}) = \int_0^\infty e^{-2x} \cdot 3e^{-3x}\,dx = \int_0^\infty 3e^{-5x}\,dx$$

$$= \left[-\frac{3}{5} e^{-5x}\right]_0^\infty = \frac{3}{5}.$$

So $\mathrm{Var}(e^{-2X}) = \frac{3}{7} - \left(\frac{3}{5}\right)^2 = \frac{3}{7} - \frac{9}{25} = \frac{75-63}{175} = \frac{12}{175}.$

Or (but this is far more complicated)

let $W = e^{-2X}$ then W takes values in $(0,1]$.

$$P(W \le w) = P(e^{-2X} \le w) = P(-2X \le \ln(w))$$
$$= P\left(X \ge -\tfrac{1}{2}\ln(w)\right)$$
$$= 1 - F_X\left(-\tfrac{1}{2}\ln(w)\right)$$

So
$$f_W(w) = -f_X\left(-\tfrac{1}{2}\ln(w)\right) \cdot -\frac{1}{2} \cdot \frac{1}{w}$$
$$= 3e^{3 \cdot \frac{1}{2}\ln w} \cdot -\frac{1}{2w} = 1\tfrac{1}{2}\,\frac{w^{1\frac{1}{2}}}{w} = \frac{3}{2} w^{\frac{1}{2}} \quad \text{for } w \in (0,1]$$

So
$$E(W^2) = \int_0^1 w^2 \cdot \frac{3}{2} w^{1/2} = \left[\frac{2}{7} \cdot \frac{3}{2} w^{7/2}\right]_0^1 = \frac{3}{7} \quad \text{and}$$
$$E(W) = \int_0^1 w \cdot \frac{3}{2} w^{1/2} = \left[\frac{2}{5} \cdot \frac{3}{2} w^{5/2}\right]_0^1 = \frac{3}{5}.$$

Same result.

For $w < 0$: $f_W(w) = 0$

For $w \geq 0$: $f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx$

$$= \int_0^{\infty} 3e^{-3x} f_Y(w-x)\, dx = \int_0^{w} 3e^{-3x} \cdot 4e^{-4(w-x)}\, dx$$

$$= \int_0^{w} 12e^{-4w+x}\, dx = 12e^{-4w}\left[e^x\right]_0^w$$

$$= 12e^{-4w}\left(e^w - 1\right) = 12e^{-3w} - 12e^{-4w}$$

2) $P_X(x) = \left(\frac{1}{2}\right)^{x-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^x$ for $x = 1, 2, \ldots$

$P_Y(y) = \frac{4^y}{y!} e^{-4}$. for $y = 1, 2, \ldots$

$$P(X=Y) = \sum_{k=1}^{\infty} P(X=k, Y=k)$$

$$\overset{\text{independence}}{=} \sum_{k=1}^{\infty} P(X=k) P(Y=k)$$

$$= \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)^k \cdot (4^k)}{k!} \quad e^{-4}$$

$$= \sum_{k=1}^{\infty} \frac{2^k}{k!} \cdot e^{-4}$$

$$= (e^2 - 1) \cdot e^{-4}.$$

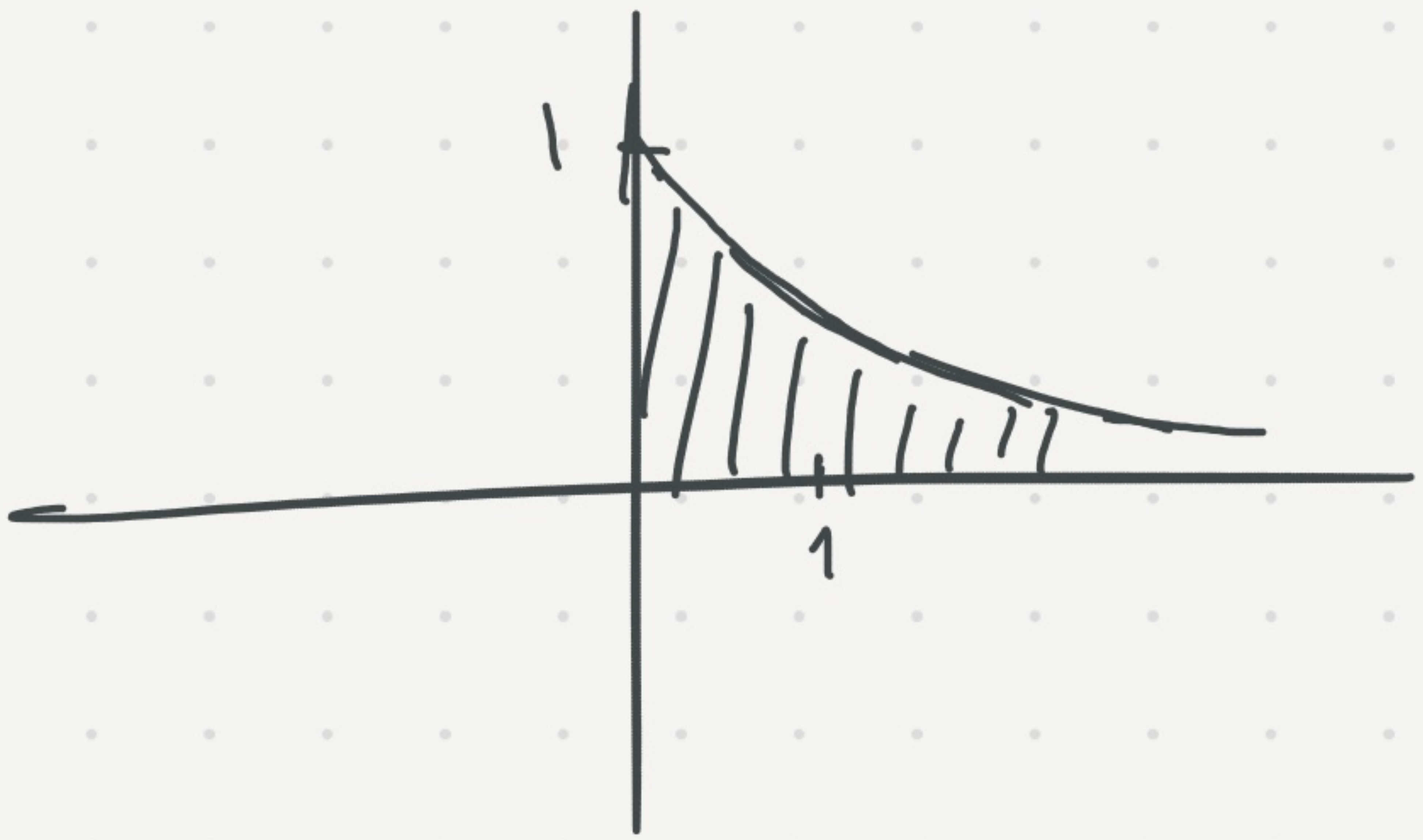
$$= e^{-2} - e^{-4}.$$

(or $\left(\sum_{k=1}^{\infty} \frac{2^k}{k!} \cdot e^{-2}\right) e^{-2}$

$= (1 - e^{-2}) \cdot e^{-2}$.

3 a) For $x \geq 0$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$



$$= \int_0^{\frac{1}{x+1}} 3(x+1)e^{-3x}\,dy =$$

$$= \left[3(x+1)e^{-3x} \cdot y\right]_0^{\frac{1}{x+1}} = 3e^{-3x},$$ so X is exponential with parameter 3.

3b) $E(Y|X=x) =$

$$\int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy = \int_0^{\frac{1}{x+1}} \frac{y \cdot 3(x+1)e^{-3x}}{3e^{-3x}}\, dy$$

$$= \int_0^{\frac{1}{x+1}} y \cdot (x+1)\, dy = \left[\frac{1}{2} y^2 \cdot (x+1)\right]_0^{\frac{1}{x+1}} = \frac{1}{2(x+1)}.$$

Or you may observe that $f_{Y|X}(y|x) = x+1$ for $0 \le y \le \frac{1}{x+1}$, so conditionally on $X=x$, Y is uniform on $\left[0, \frac{1}{x+1}\right]$. So $E(Y|X=x) = \frac{1}{2(x+1)}$.

3c) Observe that

$$\frac{1}{x+1} = \frac{1}{2}x \quad \text{if}$$

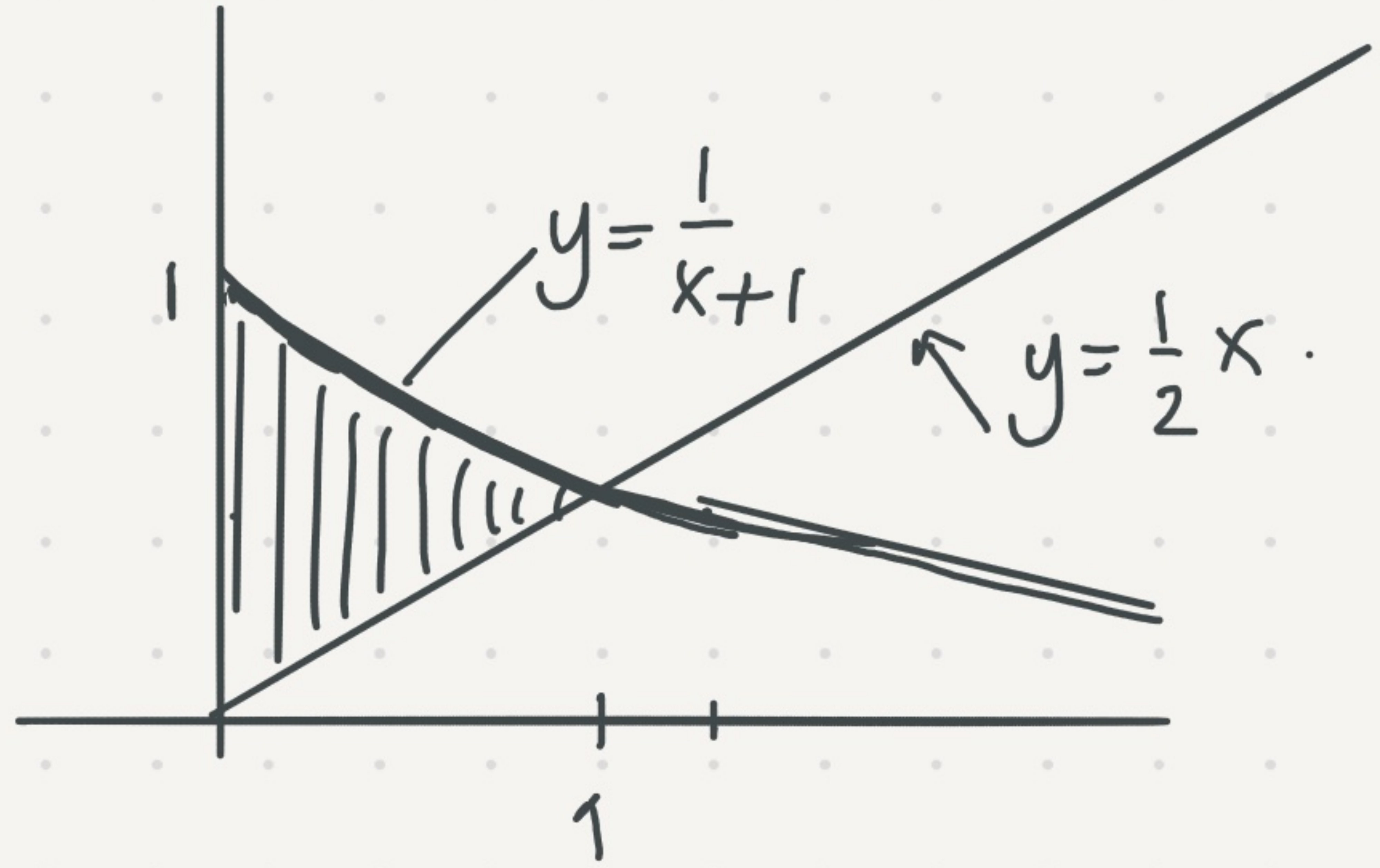
$$\frac{1}{2}x(x+1) = 1$$

$$x(x+1) - 2 = 0$$

$$x^2 + x - 2 = 0$$

$$(x-1)(x+2) = 0$$

$x = 1$ (or $x = -2$ but not relevant)



So $$P(Y > \tfrac{1}{2}X) = \int_0^1 \int_{\frac{1}{2}x}^{\frac{1}{x+1}} 3(x+1)e^{-3x}\, dy\, dx$$

$$\int_0^1 3e^{-3x}\left(1 - \tfrac{1}{2}x(x+1)\right) dx =$$

$$\int_0^1 3e^{-3x}\left(1-\frac{1}{2}x^2-\frac{1}{2}x\right)dx$$

$$= \left[-e^{-3x}\cdot\left(1-\frac{1}{2}x^2-\frac{1}{2}x\right)\right]_0^1 - \int_0^1 -e^{-3x}\left(-x-\frac{1}{2}\right)dx$$

$$= 1 + \int_0^1 - e^{-3x}\left(x+\frac{1}{2}\right)dx$$

$$= 1 + \left[\frac{1}{3}e^{-3x}\left(x+\frac{1}{2}\right)\right]_0^1 - \int_0^1 \frac{1}{3}e^{-3x}\,dx$$

$$= 1 + \frac{1}{2}e^{-3} - \frac{1}{6} + \left[\frac{1}{9}e^{-3x}\right]_0^1$$

$$= 1 + \frac{1}{2}e^{-3} - \frac{1}{6} + \frac{1}{9}e^{-3} - \frac{1}{9} = \frac{13}{18} + \frac{11}{18}e^{-3} \quad (\approx 0{,}75).$$

4 a) $P(3X > 4Y) = P(3X - 4Y > 0)$

$3X - 4Y$ is normally distributed with $\mu = 3 - 4 = -1$ and variance $3^2 \cdot 4 + 4^2 \cdot 4 = 5^2 \cdot 4 = 100$.

So $P(3X - 4Y > 0) = P\left(Z > \frac{0 - (-1)}{10}\right)$

$= P\left(Z > \frac{1}{10}\right) = 1 - \Phi\left(\frac{1}{10}\right) = 1 - 0{,}5398$
$= 0{,}4602.$

4b $\text{Cov}(X+Y, X-2Y)$

$= \text{Cov}(X,X) - 2\,\text{Cov}(X,Y) + \text{Cov}(Y,X) - 2\,\text{Cov}(Y,Y) = \text{Var}(X) - 2\,\text{Var}(Y)$

$= 4 - 8 = -4$

Since $\text{Cov}(X,Y) = 0$ by independence.

let X_i be your winnings in the ith game

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$$P(X_i = 10) = \frac{1}{10} \quad \text{and} \quad P(X_i = -2) = \frac{9}{10}$$

$$E(X_i) = 10 \cdot \frac{1}{10} - 2 \cdot \frac{9}{10} = -\frac{8}{10}.$$

$$E(X_i^2) = 100 \cdot \frac{1}{10} + 4 \cdot \frac{9}{10} = 13{,}6$$

$$\cancel{E}Var(X_i) = 13{,}6 - 0{,}64 = 12{,}96$$

$$P(X_1 + \cdots + X_{100} < -100) = P\left(\frac{X_1 + \cdots + X_{100} + 80}{\sqrt{100} \cdot \sqrt{12{,}96}} < \frac{-100+80}{\sqrt{1296}}\right)$$

$$\approx 1 - \Phi(0{,}56) = 1 - 0{,}7123 = 0{,}2877$$

or ≤ -104 Everything between -104 and 100 is fine but gives a slightly different result....

or let L be the number of times you lose.

Then you lose more than €100 if $L \geq 92$

L is $\mathrm{Bin}(100, 0{,}9)$

$$(92 \cdot -2 + 8 \cdot 10 = -104 \;,\; 91 \cdot -2 + 9 \cdot 10 = 92)$$

$$P(L \geq 92) \approx P\left(\frac{L - 100 \cdot 0{,}9}{\sqrt{100 \cdot 0{,}9 \cdot 0{,}1}} \geq \frac{92 - 90}{3}\right)$$

$$\approx 1 - \Phi(0{,}67) = 1 - 0{,}7486 = 0{,}2514$$

(With cont correction: $P\left(\frac{L - 100 \cdot 0{,}9}{3} \geq \frac{91{,}5 - 90}{3}\right)$

$$\approx 1 - \Phi(0{,}5) = 1 - 0{,}6915 = 0{,}3085)$$

Or at most 8 winnings.

The exact prob $P(Y \geq 92)$ is $0{,}32\ldots$ for your information. Observe that $np(1-p) = 9$, so it does not meet the condition $np(1-p) > 10$, although it is not that bad.

Same people interpreted the question as

$$P(Y \leq 51) = P\left(\frac{Y-90}{3} \leq \frac{51-90}{3}\right) \approx 0.$$