# Midterm Probability 2021

1a) $X \sim Bin(n=5,\ p=\frac{1}{6})$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$

$$= 1 - \left(\frac{5}{6}\right)^5 - \binom{5}{1}\left(\frac{5}{6}\right)^4 \cdot \frac{1}{6}$$

$$= 1 - \frac{3125}{7776} - \frac{3125}{7776} = \frac{1526}{7776} = \frac{763}{3888}$$

$$\approx 0{,}196$$

1a Or $P(X \geq 2) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= \binom{5}{2}\cdot\left(\frac{5}{6}\right)^3\cdot\left(\frac{1}{6}\right)^2 + \binom{5}{3}\cdot\left(\frac{5}{6}\right)^2\cdot\left(\frac{1}{6}\right)^3 + \binom{5}{4}\frac{5}{6}\left(\frac{1}{6}\right)^4 + \left(\frac{1}{6}\right)^5$$

$$= 10\cdot\frac{125}{7776} + 10\cdot\frac{25}{7776} + 5\cdot\frac{5}{7776} + \frac{1}{7776} = \frac{1526}{7776}$$

$$= \frac{763}{3888} \approx 0{,}196.$$

1b) $P(\text{large straight}) = \dfrac{\#\text{ ways to get a large straight}}{\text{total number of possible outcomes}}$

$$= 2 \cdot \frac{5!}{6^5} = \frac{240}{7776} = \frac{5}{162} \approx 0{,}031$$

Or $P(\text{we obtain } 1,2,3,4 \text{ and } 5) + P(\text{we obtain } 2,3,4,5 \text{ and } 6)$

$$= \frac{5!}{6^5} + \frac{5!}{6^5} = \frac{5}{162} \approx 0{,}031.$$

2a Let $M_i$ be the event that the ball with label $i$ is moved from urn 1 to urn 2,

$$P(X=1) = P(M_1)P(X=1 \mid M_1) = \frac{1}{3}\cdot\frac{1}{5} = \frac{1}{15}$$

$$P(X=2) = P(M_2)P(X=2 \mid M_2) + \underbrace{P(M_1)P(X=2 \mid M_1) + P(X=2)P(X=2 \mid M_3)}_{\text{or } P(M_1 \cup M_3)P(X=2 \mid M_1 \cup M_3)}$$

$$= \frac{1}{3}\cdot\frac{2}{5} + \frac{1}{3}\cdot\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5} = \frac{4}{15}$$

Similarly, $P(X=3) = \frac{4}{15}$

$P(X=4) = P(X=5) = \frac{1}{5}.$

Now $E(X) = \sum_{x} x\,P(X=x) = 1 \cdot \frac{1}{15} + 2 \cdot \frac{4}{15} + 3 \cdot \frac{4}{15} + 4 \cdot \frac{3}{15} + 5 \cdot \frac{1}{5}$

$= 3\frac{1}{5}$

$E(X^2) = 1^2 \cdot \frac{1}{15} + 2^2 \cdot \frac{4}{15} + 3^2 \cdot \frac{4}{15} + 4^2 \cdot \frac{1}{5} + 5^2 \cdot \frac{1}{5} = \frac{176}{15} = 11\frac{11}{15}$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2$

$= 11\frac{11}{15} - \left(3\frac{1}{5}\right)^2 = 1\frac{37}{75} \approx 1{,}49.$

(2b) $Y \sim \text{geom}\left(p = \frac{1}{7}\right)$ So $P(Y=y) = (1-p)^{y-1} \cdot p$
for $y \geq 1$

$P(3 \leq Y < 10)$

$= P(Y \leq 9) - P(Y \leq 2)$

$= (1 - P(Y > 9)) - (1 - P(Y > 2))$

$= P(Y > 2) - P(Y > 9)$

$= \left(\frac{6}{7}\right)^2 - \left(\frac{6}{7}\right)^9 \approx 0{,}48$

first nine are failures.

first two are failures

(exact answer $\dfrac{6^2 \cdot 7^7 - 6^9}{7^9} = \dfrac{19569852}{40353607}$)

2b Alternative

$$P(3 \leq Y < 10) = \sum_{k=3}^{9} P(Y=k)$$

$$= \sum_{k=3}^{9} \left(\frac{6}{7}\right)^{k-1} \cdot \frac{1}{7} = \frac{\left(\frac{6}{7}\right)^2 \cdot \frac{1}{7} - \left(\frac{6}{7}\right)^9 \cdot \frac{1}{7}}{1-\frac{6}{7}} =$$

$$= \left(\frac{6}{7}\right)^2 - \left(\frac{6}{7}\right)^9$$

$=$

3 a $P(Q \cup S) = P(Q) + P(S) - P(Q \cap S)$

$= P(Q) + P(H)P(S|H) + P(L)P(S|L)$
$+ P(Q)P(S|Q) - P(Q)P(S|Q)$

$= 0{,}25 + 0{,}45 \cdot 0{,}03 + 0{,}3 \cdot 0{,}1 = 0{,}2935$

Or $P(Q \cup S) = P(Q) + P(H \cap S) + P(L \cap S)$
$= P(Q) + P(H)P(S|H) + P(L)P(S|L)$
$= 0{,}25 + 0{,}45 \cdot 0{,}03 + 0{,}3 \cdot 0{,}1 = 0{,}2935$

3b) $$P(H|I) = \frac{P(H \cap I)}{P(I)}$$

$$= \frac{P(H)P(I|H)}{P(H)P(I|H) + P(L)P(I|L) + P(Q)P(I|Q)}$$

$$= \frac{0{,}45 \cdot 0{,}07}{0{,}45 \cdot 0{,}07 + 0{,}3 \cdot 0{,}05 + 0{,}25 \cdot 0{,}06}$$

$$= \frac{0{,}0315}{0{,}0615} \approx 0{,}51.$$

3c L and G are independent if $P(L \cap G) = P(L)P(G)$

$$P(L \cap G) = P(L) P(G|L) = 0{,}3 \cdot 0{,}85$$

$$P(L) = 0{,}3$$

$$P(G) = P(H)P(G|H) + P(L)P(G|L) + P(Q)P(G|Q)$$

$$= 0{,}45 \cdot 0{,}9 + 0{,}3 \cdot 0{,}85 + 0{,}25 \cdot 0{,}76 = 0{,}85.$$

So yes, L and G are independent.

(Also fine if you observe that $P(G|L) = P(G)$ or that $P(L|G) = P(L)$)

4 a) $\int_{-\infty}^{\infty} f_X(x) = 1$ gives

$$\int_0^1 c(1-x)x^2\,dx = c\int_0^1 (x^2 - x^3)\,dx$$

$$= c\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{c}{12} = 1 \quad \text{So } c = 12$$

4b) $F_X(x) = 0$ for $x \le 0$

For $0 < x < 1$: $F_X(x) = \int_0^x 12(1-t)t^2\,dt$

$$= \left[4t^3 - 3t^4\right]_0^x = 4x^3 - 3x^4$$

$F_X(x) = 1$ for $x > 1$.

4c) $E(X^2 + X) = \int_{-\infty}^{\infty} (x^2 + x) f_X(x)\, dx$

$= \int_0^1 (2x^2 + x)(12x^2 - 12x^3)\, dx$

$= \int_0^1 (24x^4 - 24x^5 + 12x^3 - 12x^4)\, dx$

$= \left[ \frac{12}{5} x^5 - 4x^6 + 3x^4 \right]_0^1 = \frac{7}{5}$

5a) $P(|X-\mu| > 2\frac{1}{2}\sigma) =$

$$= P(X-\mu > 2\tfrac{1}{2}\sigma) + P(X-\mu < -2\tfrac{1}{2}\sigma)$$

$$= P\left(\frac{X-\mu}{\sigma} > 2\tfrac{1}{2}\right) + P\left(\frac{X-\mu}{\sigma} < -2\tfrac{1}{2}\right)$$

$$= 2\cdot(1-\Phi(2{,}5)) =$$

$$= 2(1-\Phi(2{,}5)) = 2(1-0{,}9938) = 0{,}0124$$

5b $P(X>c) = P\left(\frac{X-1}{2} > \frac{c-1}{2}\right) =$

$1 - \Phi\left(\frac{c-1}{2}\right).$

So $1 - \Phi\left(\frac{c-1}{2}\right) = 0{,}975$ gives

$\Phi\left(\frac{c-1}{2}\right) = 0{,}025$; so

$\Phi\left(\frac{1-c}{2}\right) = 0{,}975 = \Phi(1{,}96)$ → or conclude that $\frac{c-1}{2} = -\Phi^{-1}(0{,}975) = -1{,}96$

So $\frac{1-c}{2} = 1{,}96$ ; $1-c \approx 3{,}92$ $c \approx -2{,}92$.