Exam Probability Theory

June 30, 2020, 8.45-11.30

- This exam consists of six questions and a table. You can score 45 points. Your grade is given by (5+number of points)/5.
- You may use a simple calculator, but a graphical or programmable calculator is not allowed.
- Explain your answers clearly.
- You have to keep track of the time yourself. Make sure that you stop at 11.30 (or 30 minutes later if you have extra time) and upload your answers as one single pdf via Canvas within 10 minutes.
- If you want to stop before the end time of the exam, ask permission of your TA via the chat. Please only do this if you would have to wait more than 15 minutes until the end time of the exam. After the TA gives you permission, upload your answers as one single pdf via canvas within 10 minutes and additionally let know to the TA in the chat when you have finished uploading. You can then leave the meeting, but it is not allowed to have any communication with other students about the exam before 12.00.
- If you need to go to the bathroom, let the TA know in the chat and wait for permission. If you don't get permission within 2 minutes, just go and mention when you are back.
- If Corrie Quant wants to contact you regarding your exam, she will send you an email before Thursday July 2nd 10.00. Please check your VUmail on Thursday July 2nd in the morning.
- Please start each of the six exercises on a new page!
- Good luck!

1. A fair die is thrown twice. Let X be the maximum of the two outcomes.
(a) [3 points] Compute $E(X)$.
(b) [3 points] Draw the cumulative distribution function $F_X(x)$ of X.
(c) [3 points] Compute the conditional probability that one of the outcomes is 'two', given that $X = 5$.

2. At a political meeting there are 4 liberals and 3 conservatives. We choose three people uniformly at random to form a committee. Let A be the event that there is at least one liberal in the committee and B the event that there are more liberals in the committee than conservatives.
(a) [3 points] Compute $P(A)$.
(b) [3 points] Compute $P(A^c \cup B)$.

3. [3 points] A gambler has a fair coin and a biased coin in his pocket. The biased coin shows heads with probability $\frac{3}{4}$ and tails with probability $\frac{1}{4}$. He selects one of the coins at random (so each of the coins has probabily $\frac{1}{2}$ to be chosen) and flips it three times. Let X be the number of times he obtains heads. Compute the conditional probability that he selected the biased coin, given that $X = 2$.

4. Let X be a random variable with density function

$$f_X(x) = \begin{cases} \frac{1}{8}(x+1) & \text{if } -1 \leq x \leq 3, \\ 0 & \text{otherwise,} \end{cases}$$

(a) [3 points] Compute the variance of X.
(b) [3 points] Compute the density $f_Y(y)$ of the random variable $Y := X^2$. Distinguish between the situations $y \leq 0$, $0 < y \leq 1$, $1 < y \leq 9$ and $y > 9$.

5. Suppose that X and Y are jointly continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} xe^{-x(1+y)} & \text{if } x \geq 0 \text{ en } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [2 points] Show that X is exponentially distributed with parameter 1.
(b) [2 points] Show that the marginal density function of Y is given by

$$f_Y(y) = \begin{cases} \frac{1}{(1+y)^2} & \text{if } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(c) [2 points] Are X and Y independent (don't forget to explain your answer!).
(d) [3 points] Compute $E(X|Y = y)$ for $y \geq 0$.

Please turn over!

6 Let $X_1, X_2, X_3, \ldots, X_{100}$ be independent exponential random variables with parameter 1. Recall that this implies that $E(X_i) = 1$ and $\text{Var}(X_i) = 1$.
(a) [3 points] Use the Central Limit Theorem to approximate

$$P(97.5 < X_1 + \cdots + X_{100} < 115).$$

(b) [3 points] Compute $P(X_1 > 2X_2)$.
(c) [3 points] Compute $\text{Cov}(X_1, X_1 - 4X_2)$.
(d) [3 points] Compute the density function of $Z := X_1 + X_2$.

Table

Area ϕ(x) under the standard normal curve to the left of x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998