# Probability Theory June 30.

## Exercise 1

1) $\Omega = \{(1,1), (1,2), \dots, (6,6)\}$

$P(X=1) = P(\{(1,1)\}) = \frac{1}{36}$

$P(X=2) = P(\{(1,2), (2,1), (2,2)\} = \frac{3}{36}$

$P(X=3) = P(\{(1,3), (3,1), (2,3), (3,2), (3,3)\}) = \frac{5}{36}$

Similarly, $P(X=4) = \frac{7}{36}$, $P(X=5) = \frac{9}{36}$,

$P(X=6) = \frac{11}{36}$.

So $E(X) = \sum_x x P(X = x)$

$$= 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36}$$

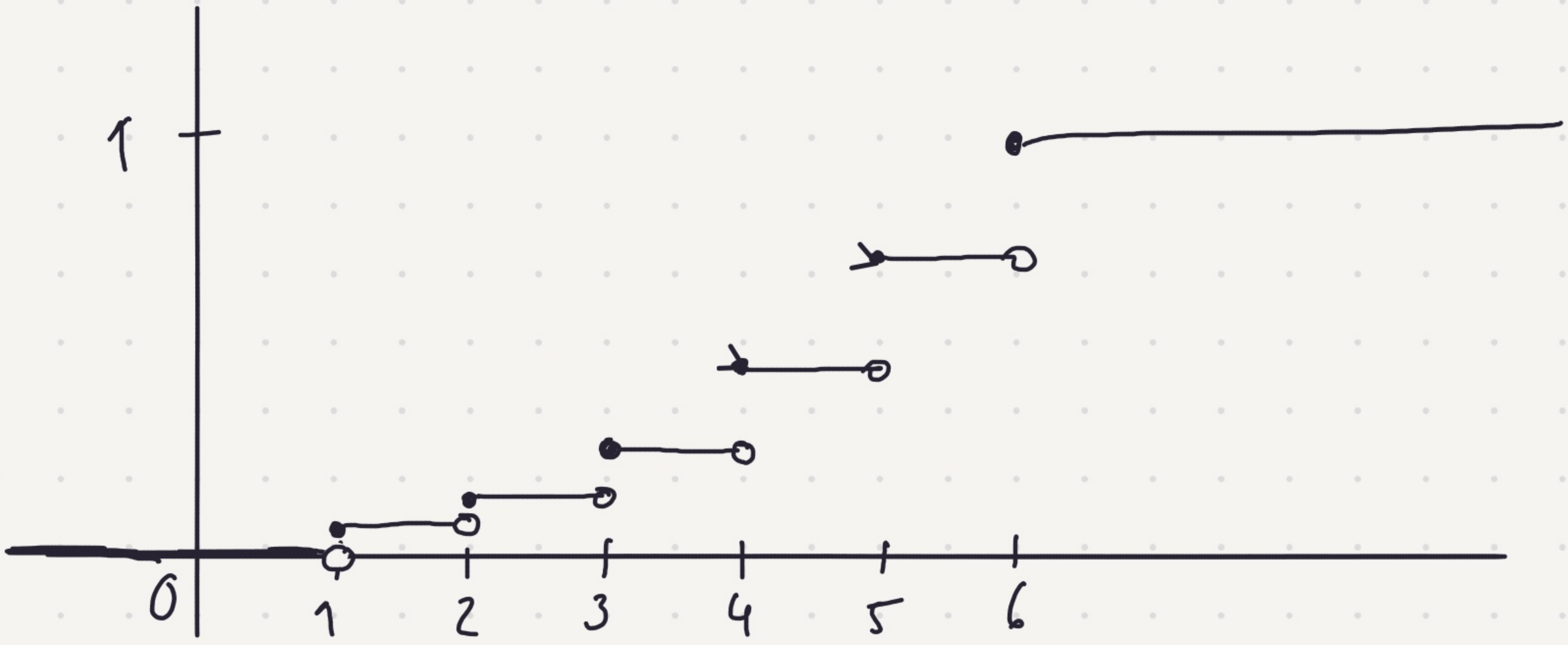
$$= \frac{161}{36} = 4\frac{17}{36}.$$

b) Observe that

$F_X(1) = P(\text{both values} \leq 1) = \frac{1}{36}$

$F_X(2) = P(\text{both values} \leq 2) = \frac{4}{36}$

$F_X(3) = \frac{9}{36}$; $F_X(4) = \frac{16}{36}$; $F_X(5) = \frac{25}{36}$, $F_X(6) = 1$.


1
0
1
2
3
4
5
6

c) $P(\text{one of the outcomes is '2'} \mid X=5)$

$$= \frac{P(\text{one of the outcomes '2' AND } X=5)}{P(X=5)}$$

$$= \frac{P(\{(2,5),(5,2\})}{P(X=5)} = \frac{2/36}{9/36} = \frac{2}{9}.$$

(You can also use a restricted sample space $\Omega^* = \{(1,5),(2,5),(3,5),(4,5),(5,5),(5,4),$
$(5,3),(5,2),(5,1)\}.$)

Exercise 2    4 liberals and 3 conservatives.

a) $P(A) = 1 - P(A^c) = 1 - P(\text{no liberals})$

$$= 1 - \frac{\binom{3}{3}}{\binom{7}{3}} = 1 - \frac{1}{35} = \frac{34}{35}.$$

b) Observe that if there are more liberals than conservatives, there are 2 liberals and 1 conservative or 3 liberals.

$A^c$: no liberals (so 3 conservatives).

So $A^c$ and $B$ are disjoint and

$$P(A^C \cup B) = P(A^C) + P(B)$$

$$= \frac{1}{35} + \frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}} + \frac{\binom{4}{3}}{\binom{7}{3}}$$

$$= \frac{1}{35} + \frac{6 \cdot 3}{35} + \frac{4}{35} = \frac{23}{35}.$$

Or: $P(A^C \cup B) = 1 - P((A^C \cup B)^C) = 1 - P(A \cap B^C)$

$= 1 - P(\text{one liberal and 2 conservatives})$

$$= 1 - \frac{\binom{4}{1}\binom{3}{2}}{35} = \frac{23}{35}$$

## Exercise 3

$$P(\text{biased coin} \mid X=2) = \frac{P(\text{biased coin} \cap X=2)}{P(X=2)}$$

Bayes
$$\frac{P(\text{biased coin})P(X=2 \mid \text{biased coin})}{P(\text{biased coin})P(X=2 \mid \text{biased}) + P(\text{fair})P(X=2 \mid \text{fair})}$$

$$= \frac{\frac{1}{2} \cdot \binom{3}{2} \cdot \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}}{\frac{1}{2}\binom{3}{2}\left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} + \frac{1}{2} \cdot \binom{3}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2}} = \frac{\frac{27}{128}}{\frac{27}{128} + \frac{3}{16}} = \frac{9}{17}$$

Exercise 4

(a) $Var(X) = E(X^2) - (E(X))^2$

$$E(X) = \int_{-1}^{3} x \cdot \frac{1}{8}(x+1)\, dx$$

$$= \int_{-1}^{3} \frac{1}{8}x^2 + \frac{1}{8}x \quad dx = \left[\frac{1}{24}x^3 + \frac{1}{16}x^2\right]_{-1}^{3}$$

$$= \frac{27}{24} + \frac{9}{16} + \frac{1}{24} - \frac{1}{16} = 1\frac{2}{3}.$$

$$E(X^2) = \int_{-1}^{3} x^2\left(\frac{1}{8}x+1\right) dx = \int_{-1}^{3}\left(\frac{1}{8}x^3 + \frac{1}{8}x^2\right) dx =$$

$$= \left[\frac{1}{32}x^4 + \frac{1}{24}x^3\right]_{-1}^{3} = \frac{81}{32} + \frac{27}{24} - \frac{1}{32} + \frac{1}{24} = 3\frac{2}{3}$$

So $\mathrm{Var}(X) = 3\frac{2}{3} - \left(\frac{5}{3}\right)^2 = \frac{8}{9}$.

b) We compute $F_Y(y) = P(Y \leq y)$ and then differentiate.

Observe that for $y \leq 0$ $F_Y(y) = 0$ and for $y > 9$: $F_Y(y) = 1$.

For $0 < y \leq 1$:

$$F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} < X < \sqrt{y}) =$$

$$F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

For $1 < y \leq 9$ : $F_Y(y) = P(X^2 \leq y) (= P(-\sqrt{y} \leq X \leq \sqrt{y}) =$

$P(X < \sqrt{y}) = F_X(\sqrt{y})$.

Differentiating gives that $f_Y(y) = 0$ for $y \leq 0$ , $y > 9$ ;

for $0 < y \leq 1$

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \cdot -1$$

$$= \frac{1}{2\sqrt{y}} \left( (\sqrt{y} + 1 + (-\sqrt{y} + 1) \right) = \frac{2}{2\sqrt{y}} = \frac{1}{\sqrt{y}} .$$

And for $1 < y < 9$ : $f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}} \cdot \frac{1}{8} (\sqrt{y} + 1)$

$\left(= \frac{1}{16} + \frac{1}{16\sqrt{y}}\right)$

## Exercise 5

a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

So for $x<0$: $f_X(x) = 0$ and for $x \geq 0$:

$$f_X(x) = \int_0^{\infty} x e^{-x(1+y)}\,dy = \left[e^{-x(1+y)}\right]_{y=0}^{\infty}$$

$$= e^{-x}.$$

b) For $y<0$: $f_Y(y) = 0$

For $y>0$: $f_Y(y) = \int_0^{\infty} x e^{-x(1+y)}\,dx =$

$$= \left[ x \cdot -\frac{e^{-x(1+y)}}{1+y} \right]_{x=0}^{\infty} - \int_0^{\infty} -\frac{e^{-x(1+y)}}{(1+y)}\, dx$$

$$= 0 + \left[ -\frac{e^{-x(1+y)}}{(1+y)^2} \right]_{x=0}^{x=\infty} = \frac{1}{(1+y)^2}.$$

c) No, since $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$.

(Or no, since you cannot find $g(x)$ and $h(y)$ s.t. $f_{X,Y}(x,y) = g(x)h(y)$).

d) $E(X \mid Y = y) =$

$$\int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dy$$

$$= \int_0^{\infty} x \cdot (1+y)^2 \cdot x e^{-x(1+y)}\,dx =$$

$$\left[x^2 \cdot \left(-(1+y)e^{-x(1+y)}\right)\right]_0^{\infty} + \overbrace{\int_0^{\infty} 2x \cdot (1+y) e^{-x(1+y)}\,dx}^{= 2\cdot(1+y)\cdot\text{anhwb}}$$

$$= 0 + \left[2x \cdot \left(-e^{-x(1+y)}\right)\right]_0^{\infty} + \int_0^{\infty} 2e^{-x(1+y)}\,dx$$

$$= 2 \cdot \left[-\frac{e^{-x(1+y)}}{(1+y)}\right]_0^{\infty} = \frac{2}{1+y}.$$

## Exercise 6

(a). $P(97{,}5 < X_1 + \dots + X_{100} < 115)$

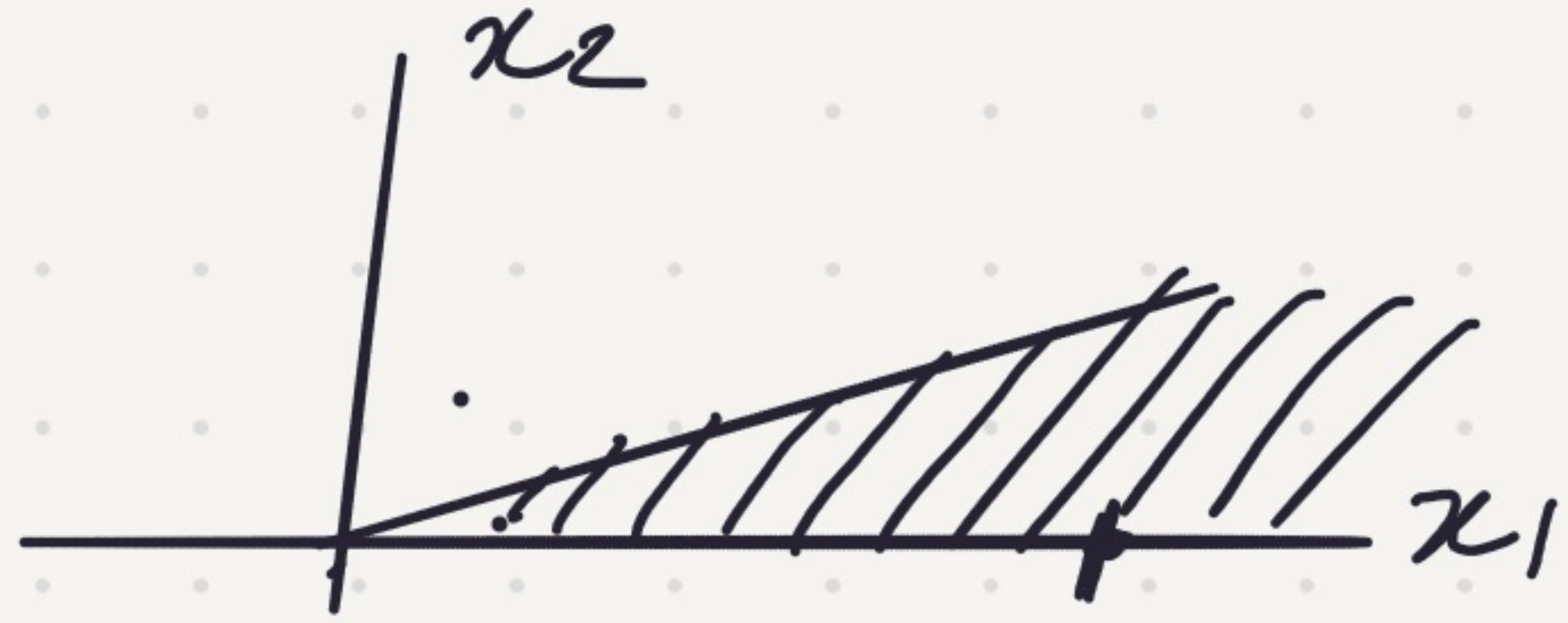
$$= P\left(\frac{97{,}5 - 100\cdot 1}{\sqrt{100}\sqrt{1}} < \frac{X_1 + \dots + X_{100} - 100}{10} < \frac{115-100}{10}\right)$$

$$\overset{CLT}{\approx} \Phi(1{,}5) - \Phi(-0{,}25)$$

$$= \Phi(1{,}5) - (1 - \Phi(0{,}25))$$

$$= 0{,}9332 - 1 + 0{,}5987 = 0{,}5319.$$

b) $P(X_1 > 2X_2)$




$$= \int_0^\infty \int_0^{1/2 x_1} f_{X_1}(x_1) f_{X_2}(x_2)\, dx_2\, dx_1$$

$$= \int_0^\infty \int_0^{\frac{1}{2}x_1} (e^{-x_1} \cdot e^{-x_2})\, dx_2\, dx_1$$

$$= \int_0^\infty \left[ -e^{-x_1} \cdot e^{-x_2} \right]_{x_2=0}^{\frac{1}{2}x_1}$$

$$= \int_0^\infty (-e^{-\frac{3}{2}x_1} + e^{-x_1})\, dx_1 = \left[ -\frac{2}{3} e^{-\frac{3}{2}x_1} + e^{-x_1} \right]_0^\infty$$

$$= -\frac{2}{3} + 1 = \frac{1}{3}.$$

Or $\int_0^\infty \int_{2x_2}^\infty e^{-x_1} e^{-x_2}\, dx_1\, dx_2$

$= \int_0^\infty \left[ - e^{-x_1} e^{-x_2} \right]_{2x_2}^\infty dx_2 = \int_0^\infty e^{-3x_2}\, dx_2$

$= \left[ -\frac{1}{3} e^{-3x_2} \right]_0^\infty = \frac{1}{3}$

bc) $\mathrm{Cov}(X_1, X_1 - 4X_2) = \mathrm{Cov}(X_1, X_1) - 4\,\mathrm{Cov}(X_1, X_2)$

$= \mathrm{Var}(X_1) - 4 \cdot 0 = 1.$

$X_1, X_2$ ind

6 d) $Z := X_1 + X_2$

for $z < 0$: $f_Z(z) = 0$

for $z \geq 0$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(z - x_1)\, dx_1$$

$$= \int_0^{\infty} e^{-x_1} \cdot f_{X_2}(z - x_1)\, dx =$$

$$= \int_0^{z} e^{-x_1} e^{-(z - x_1)}\, dx_1 = \int_0^{z} e^{-z}\, dx_1 = z e^{-z}.$$