Exam Probability Theory

May 28, 2020, 15.30-18.15

- This exam consists of six questions and a table. You can score 45 points. Your grade is given by (5+number of points)/5.
- You may use a simple calculator, but a graphical or programmable calculator is not allowed.
- Explain your answers clearly.
- You have to keep track of the time yourself. Make sure that you stop at 18.15 (or 30 minutes later if you have extra time) and upload your answers as one single pdf via Canvas within 10 minutes.
- If you want to stop before the end time of the exam, ask permission of your TA via the chat. Please only do this if you would have to wait more than 15 minutes until the end time of the exam. After the TA gives you permission, upload your answers as one single pdf via canvas within 10 minutes and additionally let know to the TA in the chat when you have finished uploading. You can then leave the meeting, but it is not allowed to have any communication with other students about the exam before 18.45.
- If you need to go to the bathroom, let the TA know in the chat and wait for permission. If you don't get permission within 2 minutes, just go and mention when you are back.
- If Corrie Quant wants to contact you regarding your exam, she will send you an email before Tuesday June 2nd 10.00. Please check your VUmail on Tuesday June 2nd in the morning.
- It is of great help if you start each of the six exercises on a new page!
- Good luck!

1. [3 points] Suppose a tutorial group of Probability Theory consists of 25 students: 10 are Mathematics students and 15 are Business Analytics students. Of the 10 Mathematics students, 4 are male and 6 are female, of the 15 Business Analytics students 10 are male and 5 are female. I select two random students from this tutorial group without replacement. Let A be the event that both students study in different programmes and let B the event that both students are male. Compute $P(A \cup B)$.

2. It is known that light bulbs produced by a certain company are defective with probability 0.01 independent of each other. The company sells these light bulbs in packages of size 5.
(a) [3 points] Mary buys a package of light bulbs. Let A be the event that she finds at least 1 defective light bulb and let B the event that she finds exactly two defective light bulbs. Compute $P(B|A)$.
(b) [3 points] The company offers a money back guarantee if a package contains at least one defect light bulb. People who obtain such a package can return their package to the company to obtain their money back. In practice only 10% of the packages that contain at least than 1 defective bulb are returned. Furthermore 0.5% also of the packages that have no defective light bulbs are returned. Customers returning a package without defective light bulbs do not get their money back.
Given that a package is returned to the company, compute the conditional probability that a returned package leads to giving money back.
(c) [3 points] One of the employees of this company decides to test light bulbs until she finds a defective bulb. Let V be the number of bulbs she tests. Give the probability mass function and the expectation of V.

3. When a certain basketball player takes his first shot in a game, he succeeds with probability $\frac{1}{2}$. If he misses his first shot, he loses confidence and his second shot will go in with probability $\frac{1}{3}$. If he misses his first 2 shots then his third shot will go in with probability $\frac{1}{4}$. His success probability will go down further to $\frac{1}{5}$ after he misses his first 3 shots. If he misses his first 4 shots, then the coach will remove him from the game. Assume that the player keeps shooting until he succeeds or he is removed from the game. Let X denote the number of shots *he misses* until his first success or until he is removed from the game.
(a) [3 points] Compute $E(X)$.
(b) [2 points] Draw the cumulative distribution function $F_X(x)$ of X.

Please turn over!

4. An urn contains 10 balls of which 2 are white, 3 are yellow, and 5 are red. John takes 6 balls with replacement. Let X be the number of white balls, Y the number of yellow balls, and Z the number of red balls he obtains.
(a) [2 points] Compute $P(X = 1, Y = 2, Z = 3)$.
(b) [2 points] Are X and Y independent?
(c) [3 points] For this part you need the table. Mary uses the same urn with the 10 balls and takes 100 balls with replacement. Let W be the number of red balls she obtains. Give an approximation of $P(48 \leq W \leq 55)$ that is based on the central limit theorem.

5. Suppose that X and Y are jointly continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} 8x^2y & \text{if } 0 < x < 1 \text{ and } 0 < y < \sqrt{x} \\ 0 & \text{otherwise.} \end{cases}$$

(a) [1 point] Make a sketch of the area for which the density is positive.
(b) [2 points] Show that the marginal density function of X is given by

$$f_X(x) = \begin{cases} 4x^3 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(c) [3 points] Compute $E(Y|X = x)$, for $0 < x < 1$.
(d) [3 points] Compute $P(Y > X)$.
(e) [3 points] Suppose that it is given that $E(X) = \frac{4}{5}$; $E(X^2) = 2/3$; $E(Y) = \frac{16}{27}$; $E(Y^2) = \frac{2}{5}$ and $E(XY) = \frac{16}{33}$. Compute the correlation of X and Y, round your answer to three decimals.

6. Let X and Y be independent continuous random variables with density functions:

$$f_X(x) = \begin{cases} \frac{1}{8}x & \text{if } 0 < x < 4, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} e^{-y} & \text{if } y > 0, \\ 0 & \text{otherwise,} \end{cases}$$

respectively.
(a) [3 points] Compute the variance of X.
(b) [3 points] Compute the density function $f_W(w)$ of $W := |X - 2|$.
(c) [3 points] Compute the density function $f_Z(z)$ of $Z := X + Y$ for $0 < z < 4$.

Table

Area ϕ(x) under the standard normal curve to the left of x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998