Exam probability Theory
May 28, 2020

Exercise 1

25 students
- 10 Math
  - 4 male
  - 6 female
- 15 BA
  - 10 male
  - 5 female

Select 2 random students.

A: both in different programmes
B: both male

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- $P(A) = \dfrac{10 \cdot 15}{\binom{25}{2}} = \dfrac{150}{300} = \dfrac{1}{2}.$

(or $P(A) = \dfrac{10}{25} \cdot \dfrac{14}{24} \cdot 2$, or

$P(A) = 1 - \dfrac{\binom{10}{2}}{\binom{25}{2}} - \dfrac{\binom{15}{2}}{\binom{25}{2}}$)

- $P(B) = \dfrac{\binom{14}{2}}{\binom{25}{2}} = \dfrac{91}{300}$

or $P(B) = \dfrac{14 \cdot 13}{25 \cdot 24}$ etc.

$$P(A \cap B) = P(1 \text{ male math} + 1 \text{ male BA})$$
$$= \frac{4 \cdot 10}{\binom{25}{2}} = \frac{40}{300}$$

So $P(A \cup B) = \frac{1}{2} + \frac{91}{300} - \frac{40}{300} = \frac{67}{100}$.

2

a) $P(B|A) = \dfrac{P(B\cap A)}{P(A)}$

$P(A\cap B) = P(B) = \binom{5}{2}\cdot(0{,}01)^2\cdot 0{,}99^3$

$P(A) = 1 - P(A^C) = 1 - P(\text{all working})$

$= 1-(0{,}99)^5.$

So $P(B|A) = \dfrac{\binom{5}{2}(0{,}01)^2\cdot(0{,}99)^3}{1-(0{,}99)^5} (\approx 0{,}0198)$

Since the number of defect light bulbs is $\mathrm{Bin}(5, 0{,}01)$.

$P(A)$ can also be computed as

$\sum_{i=1}^{5} P(i \text{ defective})$.

b) P(at least one defective light bulb)

$$= P(A) = 1 - 0{,}99^5 \approx 0{,}049$$

Let R be the event that a package is returned.

$$P(\text{money back} \mid R) = P(A \mid R)$$

$$= \frac{P(A \cap R)}{P(R)}$$

$$\overset{\text{Bayes}}{=} \frac{P(A) \cdot P(R \mid A)}{P(A)P(R \mid A) + P(A^c)P(R \mid A^c)}$$

$$= \frac{(1 - 0{,}99^5) \cdot 0{,}1}{(1 - 0{,}99^5) \cdot 0{,}1 + 0{,}99^5 \cdot 0{,}005}$$

$$(\approx 0{,}508).$$

c) $V \sim \text{geom}(0{,}01)$

So $P(V = v) = 0{,}99^{v-1} \cdot 0{,}01$

(for $v = 1, 2, \ldots$)

$$E(V) = \frac{1}{0{,}01} = 100.$$

3) $P(X=0) = P(\text{first shot succeeds})$

$$= \frac{1}{2}$$

$P(X=1) = P(\text{first failure, } 2^{nd} \text{ succeeds})$

$$= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$P(X=2) = P(\text{first failure, } 2^{nd} \text{ failure third success})$

$$= \frac{1}{2} \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

$P(X=3) = P(\text{failure, failure, failure, success})$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} = \frac{1}{20}$$

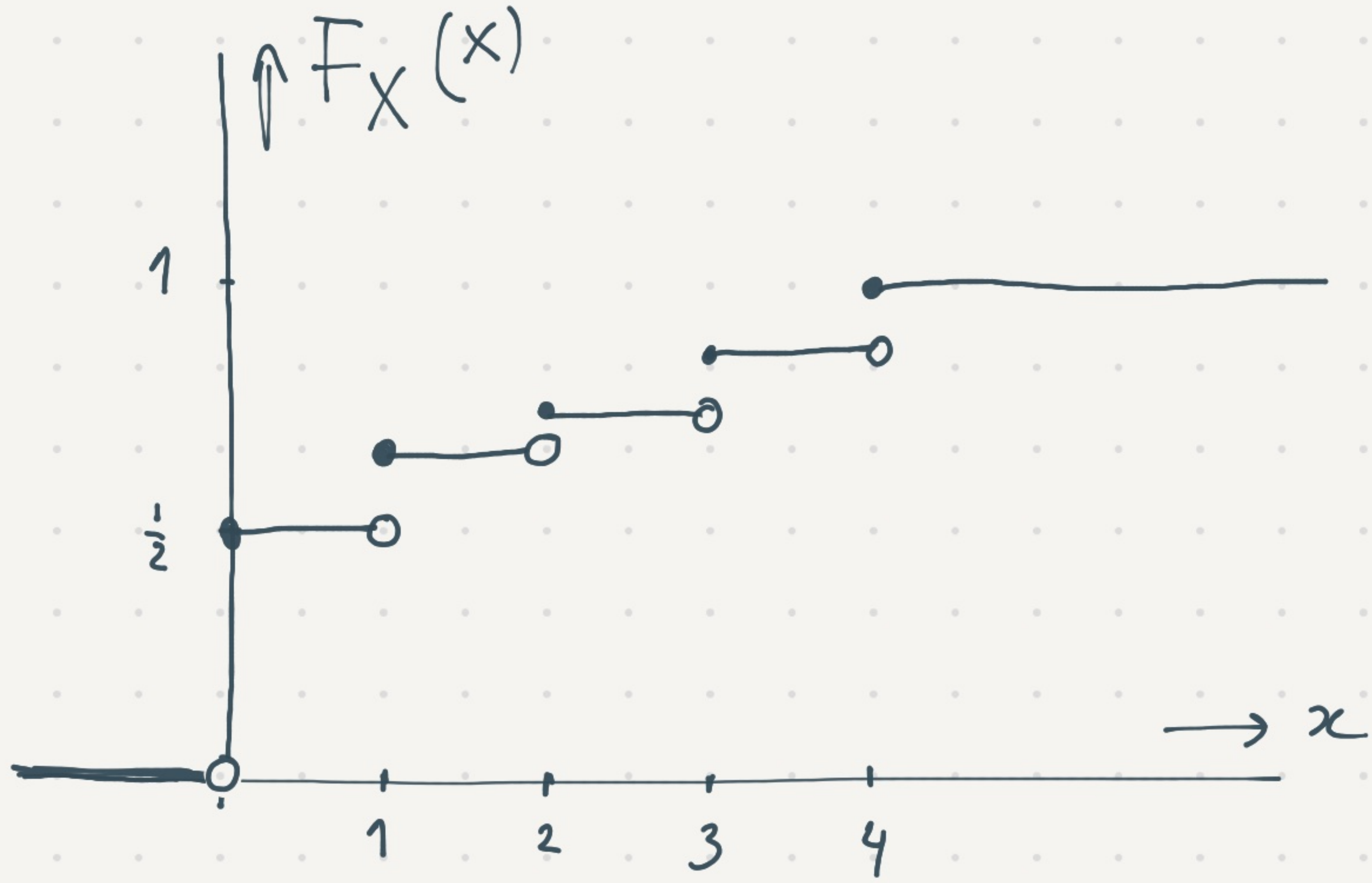
$P(X=4) = P(\text{failure, failure, failure, failure})$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5}$$

$$E(X) = \sum_{x} x P(X=x)$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{12} + 3 \cdot \frac{1}{20} + 4 \cdot \frac{1}{5}$$

$$= 1 \frac{17}{60}$$



$F_X(x)$
1
$\frac{1}{2}$
x
1
2
3
4

4

2 white
3 yellow
5 red

(a) $P(X=1, Y=2, Z=3)$

$$= \binom{6}{1,2,3} \cdot \left(\frac{2}{10}\right)^1 \cdot \left(\frac{3}{10}\right)^2 \cdot \left(\frac{5}{10}\right)^3$$

$$= \frac{6!}{2!\,3!} \cdot \frac{1}{5} \cdot \left(\frac{3}{10}\right)^2 \left(\frac{1}{2}\right)^3$$

$$= 0{,}135.$$

(b) No. If X and Y would be independent, it should have been true that $P(X=x, Y=y) = P(X=x)\,P(Y=y)$ for all x and y.

This is not the case.
Observe that $P(X=10, Y=10) = 0$ but $P(X=10) \neq 0$ and $P(Y=10) \neq 0$.

4c $W \sim Bin\left(n = 100, p = \frac{1}{2}\right)$

CLT says that $\dfrac{W - \frac{1}{2} \cdot 100}{\sqrt{100 \frac{1}{2} \cdot \frac{1}{2}}} = \dfrac{W - 50}{5}$

approximately standard normal.

$\left(np(1-p) = 100 \frac{1}{2} \cdot \frac{1}{2} = 25 > 10\right)$

$P(48 \leq W \leq 55) =$ continuity correction.

$P(47{,}5 \leq W \leq 55{,}5) =$

$= P\left(\dfrac{47{,}5 - 50}{5} \leq \dfrac{W - 50}{5} \leq \dfrac{55{,}5 - 50}{5}\right)$

$\approx \Phi(1{,}1) - \Phi(-0{,}5)$

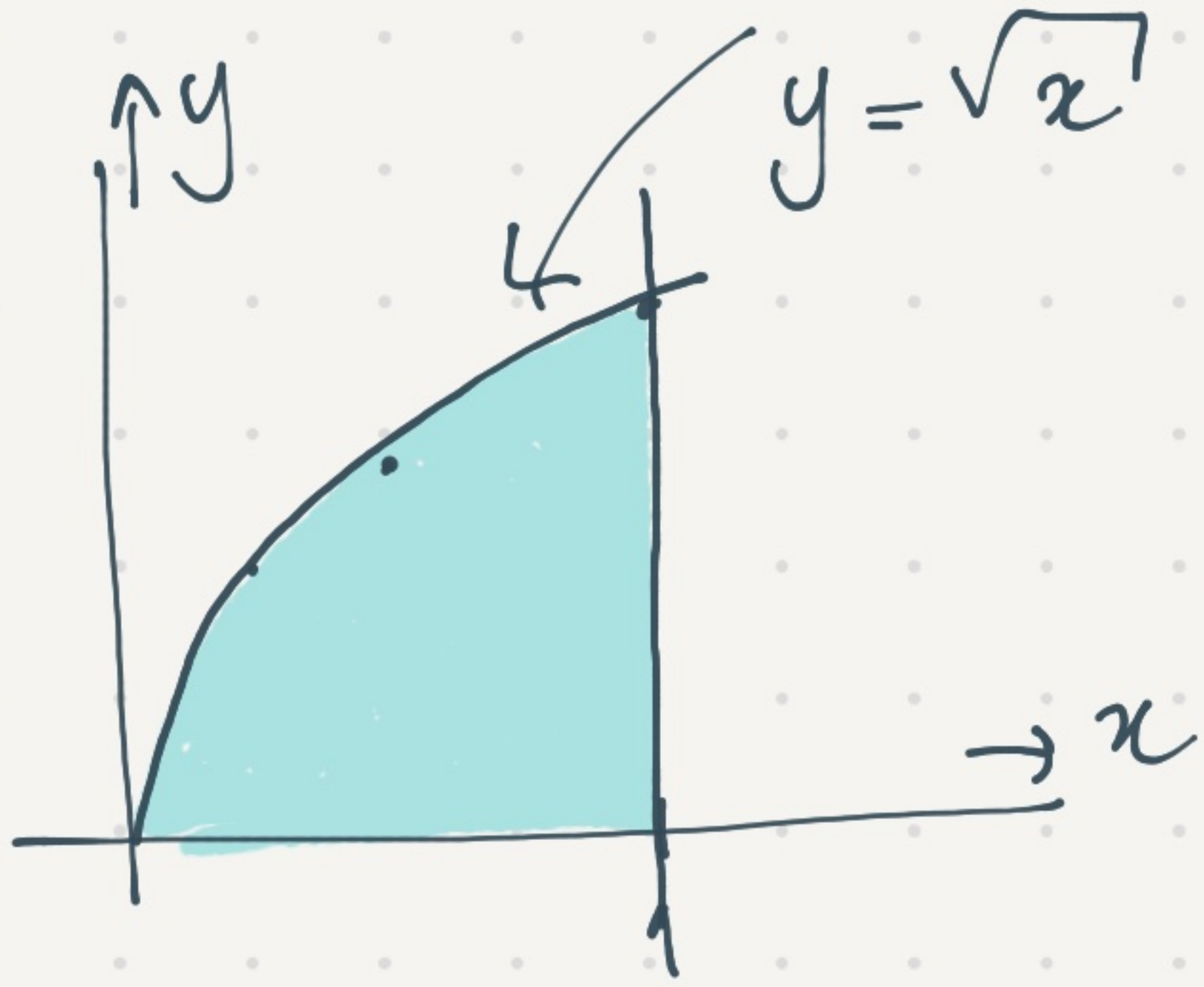
$= \Phi(1{,}1) - (1 - \Phi(0{,}5))$

$= 0{,}8643 - (1 - 0{,}6915)$

$= 0{,}5558$

5 $f_{X,Y}(x,y) = \begin{cases} 8x^2y & \text{if } \begin{array}{l} 0<x<1 \\ 0<y<\sqrt{x} \end{array} \\ 0 & \text{otherwise} \end{cases}$

a



b for $0<x<1$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_0^{\sqrt{x}} 8x^2y\,dy$$

$$= \left[4x^2y^2\right]_0^{y=\sqrt{x}} = 4x^3$$

For $x \notin (0,1)$: $f_X(x) = 0$.

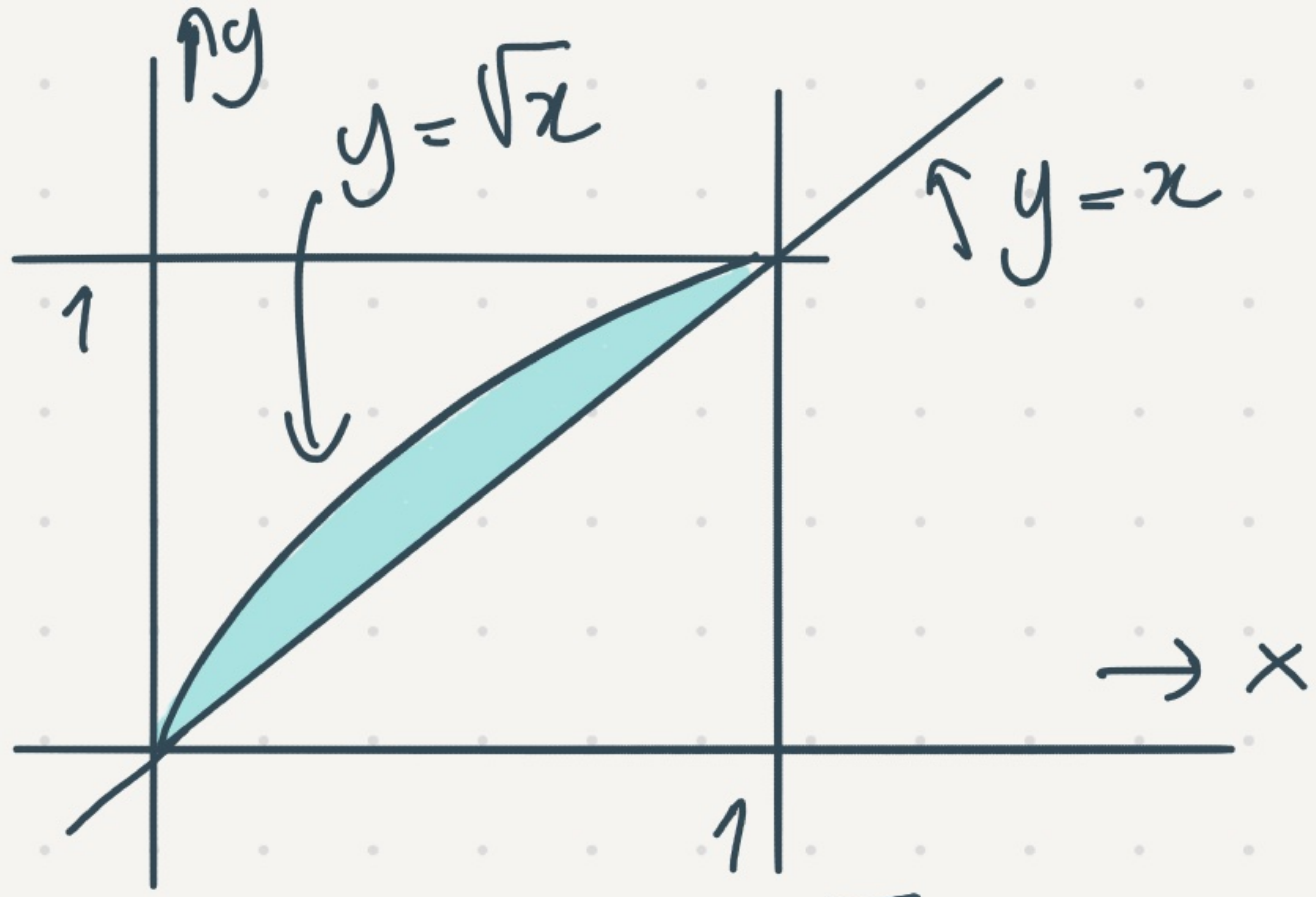
(c) For $0 < x < 1$

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y \, f_{Y|X}(y|x)\,dy$$

$$= \int_0^{\sqrt{x}} y \cdot \frac{8x^2 y}{4x^3}\,dy = \int_0^{\sqrt{x}} \frac{2y^2}{x}\,dy$$

$$= \left[\frac{2}{3}\frac{y^3}{x}\right]_0^{\sqrt{x}} = \frac{2}{3}\sqrt{x}.$$

(d)



$$P(Y>X) = \int_0^1 \int_x^{\sqrt{x}} 8x^2 y \, dy \, dx$$

$$= \int_0^1 \left[4x^2y^2 \right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 (4x^3 - 4x^4) \, dx = \left[x^4 - \frac{4}{5}x^5 \right]_0^1$$

$$= \frac{1}{5}$$

$\left(\text{or } 1 - P(Y<X) \text{ or } \int_0^1 \int_{y^2}^{y} 8x^2 y \, dx \, dy \right)$

(e) $\mathrm{Corr}(X,Y) = \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{16}{33} - \frac{4}{5} \cdot \frac{16}{27} \quad (\approx 0{,}01077441)$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75}$$

$$\approx 0{,}02667$$

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2 = \frac{2}{5} - \left(\frac{16}{27}\right)^2 = \frac{178}{3645}$$

$$\approx 0{,}04883$$

$$\mathrm{Corr}(X,Y) = \frac{\frac{16}{33} - \frac{4}{5} \cdot \frac{16}{27}}{\sqrt{2/75}\ \sqrt{178/3645}}$$

$$\approx 0{,}299$$

(6) (a) $Var(X) = E(X^2) - (E(X))^2$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_0^4 x \cdot \frac{1}{8} x \, dx = \left[\frac{1}{24} x^3\right]_0^4$$

$$= 2\tfrac{2}{3}$$

$$E(X^2) = \int_0^4 x^2 \cdot \frac{1}{8} x \, dx$$

$$= \left[\frac{1}{32} x^4\right]_0^4 = 8$$

So

$$Var(X) = 8 - \left(2\tfrac{2}{3}\right)^2 \quad \left(= \frac{8}{9}\right)$$

(b) First compute $F_W(w)$, then differentiate.

For $w \leq 0$: $F_W(w) = 0$

For $w \geq 2$: $F_W(w) = 1$

For $0 < w < 2$:

$F_W(w) = P(W \leq w) =$

$= P(|X-2| \leq w) = P(-w \leq X-2 \leq w)$

$= P(2-w \leq X \leq 2+w)$

$= F_X(2+w) - F_X(2-w)$.

So $f_W(w) = 0$ for $w \leq 0$ and for $w \geq 2$

and for $0 < w < 2$

$f_W(w) = f_X(2+w) - f_X(2-w) \cdot (-1)$

$= \frac{1}{8}(2+w) + \frac{1}{8}(2-w) = \frac{1}{2}$

(So W is uniform on $(0,2)$)

You could also work out:

for $0 < w < 2$

$$F_W(w) = P(2-w \leq X \leq 2+w)$$

$$= \int_{2-w}^{2+w} \frac{1}{8} x \, dx = \left[\frac{1}{16} x^2 \right]_{2-w}^{2+w}$$

$$= \frac{1}{16}\left(4 + 4w + w^2\right) - \frac{1}{16}\left(4 - 4w + w^2\right)$$

$$= \frac{1}{2} w.$$

So for $w \in (0,2)$ $f_W(w) = \frac{1}{2}$.

(c) Use the convolution formula

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$= \int_0^4 \frac{1}{8} x \cdot f_Y(z-x)\, dx.$$

For $0 < z < 4$:

$$f_Z(z) = \int_0^z \frac{1}{8} x \cdot e^{-(z-x)} dx$$

$$\underset{\text{P.I.}}{=} \left[\frac{1}{8} x \cdot e^{-(z-x)} \right]_0^z - \int_0^z \frac{1}{8} \cdot e^{-(z-x)} dx$$

$$= \frac{1}{8} z - \left[\frac{1}{8} e^{-(z-x)} \right]_{x=0}^{z}$$

$$= \frac{1}{8} z - \frac{1}{8} + \frac{1}{8} e^{-z}.$$

Or $f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z-y)\, dy$

$$= \int_0^{\infty} e^{-y} \cdot f_X(z-y)\, dy$$

$$= \int_0^{z} e^{-y} \cdot \frac{1}{8}(z-y)\, dy$$

$$= \Big[\frac{1}{8}(z-y) \cdot (-e^{-y}) \Big]_{y=0}^{y=z}$$

$$- \int_0^{z} -\frac{1}{8} \cdot (-e^{-y})\, dy$$

$$= +\frac{1}{8} z \quad + \Big[\frac{1}{8} e^{-y}\Big]_0^z$$

$$= \frac{1}{8} z + \frac{1}{8} e^{-z} - \frac{1}{8}.$$