Final Exam Probability Theory

July 2 2019, 8.45-11.30

- This exam consist of seven exercises and a table. You can obtain 45 points. Your grade is given by (5+number of points)/5.
- You may use a simple calculator, but it is not allowed to use a graphical or a programmable calculator.
- Explain your answers clearly!

1. Four balls are randomly drawn (without replacement) from an urn that contains 2 yellow, 3 red and 4 green balls. Let X be the number of different colours.
(a) [4 points] Show that $P(X = 1) = \frac{1}{126}$ and that $P(X = 3) = \frac{4}{7}$.
(b) [3 points] Compute the variance of X.
(c) [3 points] Let A be the event that no yellow balls are drawn and B the event that no red balls are drawn. Compute $P(A \cup B)$.

2. Suppose that 8% of men and 0.25% of women are colour blind. We consider a group consisting of 8 women and 2 men.
(a) [3 points] Compute the probability that at least one of the persons in this group is colour blind.
(b) [4 points] Suppose I tell you that exactly one of the persons in this group is colour blind. Given this information, compute the conditional probability of this person being female.

3. [4 points] Let X be a standard normal random variable. Recall that the density of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ for all } x \in \mathbb{R}.$$

Compute the density function of $Y := X^2$.

4. [4 points] We throw a fair coin 3 times. Let A be the event that we see at most one head and B the event that we see at least one head and at least one tail. Are A and B independent?

5. Let X and Y be continuous random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(x+y) & \text{if } 0 \le y \le 2 \text{ and } 0 \le x \le y, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Show that the marginal density function of Y is given by

$$f_Y(y) = \begin{cases} \frac{3}{8}y^2 & \text{if } 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(b) [4 points] Show that $E(X|Y=y) = \frac{5}{9}y$, for $0 < y \le 2$.
(c) [3 points] Compute $E(X)$ using (a) and (b).

6. Let X and Y be independent random variables: X is uniformly distributed over the interval $[-1,1]$ and Y is exponentially distributed with parameter 2 (so $f_Y(y) = 2e^{-2y}$, for $y \ge 0$).
(a) [3 points] Compute the density $f_Z(z)$ of the random variable $Z := X+Y$ for $z \in [-1,1]$.
(b) [4 points] Compute $\text{Cov}(XY, X)$.

7. [3 points] For this problem you need the table. Use the central limit theorem to approximate the probability that out of 300 independent die rolls with a single fair die, we obtain less than 90 results that are multiples of 3.