Final Exam Probability Theory

May 24 2019, 15.15-17.15

- This exam consist of six exercises and a table. You can obtain 36 points. Your grade is given by (4+number of points)/4.
- You may use a simple calculator, but it is not allowed to use a graphical or a programmable calculator.
- Explain your answers clearly!

1. Let X and Y be jointly continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-2(y-x)} & \text{if } 0 \leq x \leq 1 \text{ and } y \geq x, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [2 points] Show that X is uniformly distributed over the interval [0,1].
(b) [3 points] Compute the marginal density function $f_Y(y)$ of Y.
(c) [3 points] Show that $E(Y|X=x) = x + \frac{1}{2}$, for $0 \leq x \leq 1$.
(d) [2 points] Compute $E(Y)$ using the results of (a) and (c).
(e) [3 points] Compute the covariance between X and Y.

2. [3 points] Let X be a uniform random variable over the interval $(0,1)$ and let $Y := 1/X$. Compute the density function of Y.

3. [4 points] Let X and Y be independent random variables and assume that X is exponentially distributed with parameter 3 and that Y is exponentially distributed with parameter 2. Compute the density of $Z := \max(X,Y)$. *Hint: Observe that $Z \leq z$ if and only if both $X \leq z$ and $Y \leq z$.*

4 Let X and Y be independent continuous random variables with density functions

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 1 & \text{if } 1 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [4 points] Compute the density function $f_Z(z)$ of $Z := X + Y$.
Hint: It is convenient to distinguish between the situations $z < 1$, $1 \le z < 2$, $2 \le z < 3$ *and* $z \ge 3$.
(b) [3 points] Compute $P(Y - X > \frac{3}{2})$.
(c) [3 points] Compute $\mathrm{Cov}(2X, X - 3Y)$.

5. [3 points] Jo practises her tennis service by serving 100 times in a row. Each of her services is, independently of the previous ones, successful with probability $\frac{2}{5}$. Use the central limit theorem to approximate the probability that Jo obtains less than 50 successful services.

6. [3 points] For this exercise you need the table. Let $X_1, X_2, \ldots, X_{400}$ be independent and identically distributed continuous random variables with $E(X_i) = 1$ and $\mathrm{Var}(X_i) = 4$. Use the central limit theorem to approximate the probability that

$$P\left(\left|\frac{X_1 + \cdots + X_{400}}{400} - 1\right| \ge 0.2\right).$$

Table

Area ϕ(x) under the standard normal curve to the left of x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998