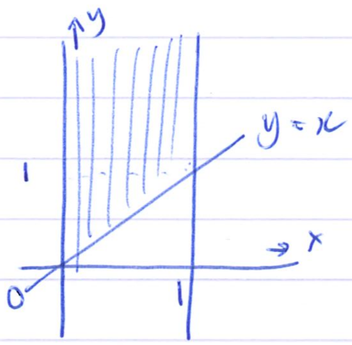
Short answers May 24 Final exam.

$$f_{X,Y}(x,y) = \begin{cases} 2\, e^{-2(y-x)} & 0 \le x \le 1 \ \& \ y \geq x \\ 0 & \text{otherwise} \end{cases}$$



(a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

So for $0 \le x \le 1$:

$$f_X(x) = \int_x^{\infty} 2\, e^{-2(y-x)}\, dy$$

$$= \left[-e^{-2(y-x)} \right]_{y=x}^{\infty} = 1$$

$f_X(x) = 0$ if $x \notin [0,1]$.

(b) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$

So for $0 \le y \le 1$:

$$f_Y(y) = \int_0^y 2\, e^{-2(y-x)}\, dx$$

$$= \left[e^{-2(y-x)} \right]_{x=0}^{\infty} = 1 - e^{-2y}$$

for $y \geq 1$: $f_Y(y) = \int_0^1 e^{-2(y-x)}\, dx$

$= \dots = e^{-2(y-1)} - e^{-2y}$.

for $y < 0$: $f_Y(y) = 0$.

(c) $E(Y|X=x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\, dy$

with $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$

So for $0 \leq x \leq 1$

$$E(Y|X=x) = \int_x^{\infty} y \cdot \frac{2e^{-2(y-x)}}{1}\, dy$$

$$= \Big[y \cdot e^{-2(y-x)} \Big]_x^{\infty} - \int_x^{\infty} -e^{-2(y-x)}\, dy$$

$$= x + \Big[-\tfrac{1}{2} e^{-2(y-x)} \Big]_{y=x}^{\infty} = x + \tfrac{1}{2}.$$

(d) $E(Y) = \int_{-\infty}^{\infty} E(Y|X=x)\, f_X(x)\, dx$

$$= \int_0^1 (x + \tfrac{1}{2}) \cdot 1\, dx = \Big[\tfrac{1}{2}x^2 + \tfrac{1}{2}x \Big]_0^1 = 1.$$

(e) $Cov(X,Y) = E(XY) - E(X)E(Y)$

$E(X) = \frac{1}{2}, \qquad E(Y) = 1$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f_{X,Y}\, dy\, dx$$

$$= \int_0^1 \int_x^{\infty} xy \cdot 2e^{-2(y-x)}\, dy\, dx$$

$$= \int_0^1 x \int_x^{\infty} y \cdot 2e^{-2(y-x)}\, dy\, dx = \text{(same as in c)}$$

$$= \int_0^1 x\left(x+\tfrac{1}{2}\right) dx = \left[\tfrac{1}{3}x^3 + \tfrac{1}{4}x^2\right]_0^1 = \frac{7}{12}$$

So $Cov(X,Y) = \frac{7}{12} - \frac{1}{2} = \frac{1}{12}$.

(2) For $y \leq 1$: $F_Y(y) = 0$
For $y > 1$: $F_Y(y) = P(Y \leq y) =$
$P(\frac{1}{X} \leq y) = P(X \geq \frac{1}{y}) = 1 - \frac{1}{y} \quad (1 - F_X(\frac{1}{y}))$

So $f_Y(y) = \frac{1}{y^2}$ for $y > 1$.

(3) For $z \geq 0$: $F_Z(z) = P(\max(X,Y) \leq z)$
$= P(X \leq z, Y \leq z) \overset{\text{ind}}{=} P(X \leq z)P(Y \leq z)$

$$= \left(\int_0^z 3e^{-3x}\, dx\right)\left(\int_0^z 2e^{-2y}\, dy\right)$$
$$= (1 - e^{-3z})(1 - e^{-2z}) = 1 - e^{-2z} - e^{-3z} + e^{-5z}.$$

So $f_Z(z) = 2e^{-2z} + 3e^{-3z} - 5e^{-5z}$ for $z \geq 0$.

(4) $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$

$$= \int_0^1 2x\, f_Y(z-x)\, dx.$$

For $z<1$ and $z \geq 3$ $f_Z(z)=0$

Observe $z-x \geq 1$ gives $x \leq z-1$

$z-x \leq 2$ gives $x \geq z-2$.

For $1 \leq z < 2$

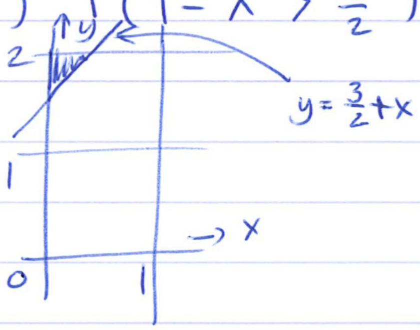
$$f_Z(z) = \int_0^{z-1} 2x \cdot 1\, dx = \left[x^2\right]_0^{z-1} = (z-1)^2.$$

For $2 \leq z < 3$

$$f_Z(z) = \int_{z-2}^{1} 2x \cdot 1\, dx = \left[x^2\right]_{z-2}^{1} = 1-(z-2)^2$$

$$= -3 - z^2 + 4z.$$

(b) $P(Y - X > \frac{3}{2})$



$$= \int_0^{\frac{1}{2}} \int_{\frac{3}{2}+x}^{2} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_0^{\frac{1}{2}} \int_{\frac{3}{2}+x}^{2} 2x\, dy\, dx$$

$$= \dots = \frac{1}{24}.$$

(c) $\mathrm{Cov}(2X, X-3Y) = 2\,\mathrm{Cov}(X,X) - 6\,\mathrm{Cov}(X,Y)$
$= 2\,\mathrm{Var}\,X - 6 \cdot 0$ (ind)
$= 2\,\mathrm{Var}(X)$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2$ gives $\mathrm{Var}(X) = \frac{1}{18}$.

(5) Let X be the number of successful services
$X \sim \mathrm{Bin}(n = 100 ; p = 0{,}4)$

$P(X < 50) = P(X \leq 49{,}5)$ (cont corr)

$= P\left(\frac{X - 100 \cdot \frac{2}{5}}{\sqrt{100 \cdot \frac{2}{5} \cdot \frac{3}{5}}} \leq \frac{49{,}5 - 40}{\sqrt{24}}\right) \overset{CLT}{\approx} \Phi(1{,}94)$

$= 0{,}9738$.

(6) $P\left(\left|\frac{X_1 + \cdots + X_{400}}{400} - 1\right) \geq 0{,}2\right.$

$= P(\cdots \geq 0{,}2) + P(\cdots \leq -0{,}2)$
$= P(X_1 + \cdots + X_{400} \geq 1{,}2 \cdot 400) + P(X_1 + \cdots + X_{400} \leq 0{,}8 \cdot 400)$

$= P\left(\frac{X_1 + \cdots + X_{400} - 400 \cdot 1}{\sqrt{400}\,\sqrt{4}} \geq \frac{480 - 400}{40}\right) + P\left(\cdots \leq \frac{320 - 400}{40}\right)$

$\overset{CLT}{\approx} 1 - \Phi(2) + \Phi(-2) = 2(1 - \Phi(2))$
$= 2 \cdot (1 - 0{,}9772)$
$= 0{,}0456$