Midterm exam March 25, 2019

- Appologies for miswritings or computational errors
- Don't look at the answers before you tried to solve the problems yourself!
- Please use the discussion section to ask questions, about this exam or material discussed earlier that is still unclear.
  It will help you and your fellow students to understand things better. Don't feel embarassed to ask a question! There are no stupid questions and we just cannot help you face to face these weeks.
  If you don't feel comfortable posting your question, you can also send a message to your TA, (see the course manual for their email addresses) and they can post both the question (without your name) and an answer.

Problem 1

Sample space: $\Omega = \{(x_1, x_2, x_3) : x_i \in \{g, f, c\}\}$, $|\Omega| = 3^3$

x_i indicates which type of player is chosen from team i.

(a) Write C for the event that the players do not form a complete team.

Observe that is easier to count $|C^c|$ than $|C|$

C^c is the event that the players form a complete team.

$$P(C) = 1 - P(C^c) = 1 - \frac{|C^c|}{|\Omega|} = 1 - \frac{3\cdot 2\cdot 1}{3^3} = \frac{21}{27} = \frac{7}{9}$$

$|C^c| = 3\cdot 2\cdot 1$ since there are 3 options for the player chosen from team 1, once you know which player is chosen from team 1, there are 2 allowed choices for the player from team 2, and once you have chosen those player, there is only one option for the player from team 3.

You can also count $|A|$ directly, but be careful not to miss anything...

(b) $P(\{(s,s,s),(g,g,g),(c,c,c)\}) = \frac{3}{3^3} = \frac{1}{9}$

(c) Observe that $A = C$ (as defined in (a)), so $P(A) = \frac{7}{9}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(B) = 1 - P(B^c) = 1 - P(\text{none of the selected players is a center})$$

$$= 1 - \left(\tfrac{2}{3}\right)^3 = \frac{19}{27}$$

$P(A \cap B) = P(\text{one player is a center, 2 players either guard or forw})$

$+ P(\text{two players are centers, other is guard or forward})$

$+ P(\text{all three players are centers})$

$$= \frac{3\cdot 2}{27} + \frac{3\cdot 2}{27} + \frac{1}{27} = \frac{13}{27}.$$

center can be from $1^{st}/2^{nd}/3^{rd}$ team

guard/forward

guard/forward

2 centers can be placed in $\binom{3}{2}$ ways

So $P(A \cup B) = \dfrac{21 + 19 - 13}{27} = 1$

You may also show this in an alternative way.

E.g. Observe that if A^c occurs, all players have a different position, so then there is at least one center

so $A^c \subset B$

So $A^c \cup A \subset A \cup B$.

Since $A^c \cup A = \Omega$, this means that $\Omega \subset A \cup B$. Since also $A \cup B \subset \Omega$ by definition of a sample space, $A \cup B = \Omega$.

Problem 2

Let F be the event that the blue coin is fair and H the event that you obtain 3 times heads. We need to compute $P(F^c | H)$

We use Bayes' formula:

You use Bayes since you need $P(F^c|H)$ while $P(H|F^c)$, $P(H|F)$ and $P(F)$ are given. That's typical for Bayes

$$P(F^c | H) = \frac{P(F^c) P(H | F^c)}{P(F^c) P(H | F^c) + P(F) P(H | F)}$$

$$= \frac{\frac{1}{5} \cdot \left(\frac{3}{4}\right)^3}{\frac{1}{5} \cdot \left(\frac{3}{4}\right)^3 + \frac{4}{5} \cdot \left(\frac{1}{2}\right)^3} = \frac{27/320}{\frac{27}{320} + \frac{1}{10}} = \frac{27/320}{59/320} = \frac{27}{59}$$

Problem 3

(a) $P(X=3) = P(X=3 \text{ and stronger team wins})$
$+ P(X=3 \text{ and weaker team wins})$
$= P(\text{stronger team wins first 3 games}) +$
$+ P(\text{weaker team wins first 3 games})$
$= 0{,}6^3 + 0{,}4^3 = 0{,}28$

$P(X=4) = P(\text{strong team wins 2 of the first 3 games and the } 4^{th})$
$+ P(\text{weak team wins 2 of the first 3 games and the } 4^{th})$
$= \binom{3}{2} \cdot 0{,}6^2 \cdot 0{,}4 \cdot 0{,}6 + \binom{3}{2} 0{,}4^2 \cdot 0{,}6 \cdot 0{,}4$
$= 0{,}3744$

$P(X=5) = 1 - P(X=3) - P(X=4)$ or
$P(X=5) = \binom{4}{2} \cdot 0{,}6^3 \cdot 0{,}4^2 + \binom{4}{2} \cdot 0{,}4^3 \cdot 0{,}6^2 = 0{,}3456$

(b) $Var(X) = E(X^2) - (E(X))^2$. $E(X) = \sum_x x \cdot P(X=x)$, so
$E(X) = 3 \cdot 0{,}28 + 4 \cdot 0{,}3744 + 5 \cdot 0{,}3456 = 4{,}0656$
$E(X^2) = \sum_x x^2 P(X=x) = 3^2 \cdot 0{,}28 + 4^2 \cdot 0{,}3744 + 5^2 \cdot 0{,}3456$
$= 17{,}1504$

So $Var(X) = 17{,}1504 - (4{,}0656)^2 \approx 0{,}62$.

(c) Suppose at most 3 games are played.
P(weaker team wins) = P(weak plays win, lose win)
+ P(weak team loses first, wins 2^{nd} and third) +
+ P(weak team wins first 2 games)
$= 0{,}4^2 \cdot 0{,}6 \cdot 2 + 0{,}4^2 = 0{,}352$.

Suppose at most 5 games are played
P(weak team wins) = P(weak wins 1^{st}, 2^{nd}, 3^{rd})
+ P(weak team loses one of the first 3, wins $4th$)
+ P(weak team loses 2 of the first 4, wins $5th$)
$= 0{,}4^3 + \binom{3}{1} \cdot 0{,}4^3 \cdot 0{,}6 + \binom{4}{2} \cdot 0{,}4^3 \cdot 0{,}6^2$
$= 0{,}31744$

So the option 2 out of 3 games is better for the weaker team.

4 a) Let X be the life time

$X \sim N(\mu = 14 \cdot 10^5, \sigma^2 = (3 \cdot 10^5)^2)$.

$$P(X < 7 \cdot 10^5) = P\left(\frac{X - 14 \cdot 10^5}{3 \cdot 10^5} < \frac{7 \cdot 10^5 - 14 \cdot 10^5}{3 \cdot 10^5}\right)$$

$$= \Phi(-2\tfrac{1}{3}) = 1 - \Phi(2\tfrac{1}{3})$$

$$\approx 1 - 0{,}9901 \approx 0{,}010.$$

standardise

If $X \sim N(\mu, \sigma)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$

Φ is the cumulative distribution of a $N(0,1)$ r.v.

4 b) I now realise that you did not yet see the Poisson distribution.

So let's compute $P(Y \leq 2)$ exactly.

$Y \sim Bin(200; p \approx 0{,}010)$

$P(Y \leq 2) = P(Y=0) + P(Y=1) + P(Y=2)$

$P(Y=0) = (1 - 0{,}010)^{200} \approx 0{,}134$

$P(Y=1) = \binom{200}{1} \cdot (0{,}990)^{199} \cdot 0{,}010 = 0{,}271$

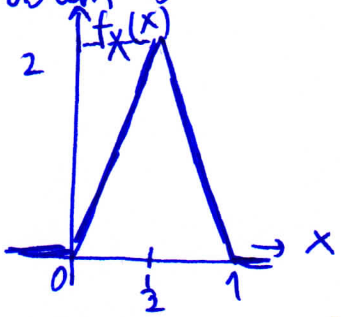
$P(Y=2) = \binom{200}{2} \cdot (0{,}990)^{198} \cdot 0{,}010^2 \approx 0{,}272$

$P(Y \leq 2) \approx 0{,}677$.

In the above computations I have used success prob. 0,010, this is not exact, since we used a table for the normal distribution.

Your calculator has a button nCr to compute binomial coefficients, or you use $\binom{200}{2} = \frac{200 \cdot 199}{2}$.

Problem 5



(a) $f_X(x) \geq 0$ for all x

$\int_{-\infty}^{\infty} f_X(x)$ must be 1. This is indeed true:

$$\int_0^{\frac{1}{2}} 4x\,dx + \int_{\frac{1}{2}}^{1} (4-4x)\,dx = \left[2x^2\right]_0^{\frac{1}{2}} + \left[4x - 2x^2\right]_{\frac{1}{2}}^{1}$$

$$= \left(\tfrac{1}{2} - 0\right) + \left(2 - \tfrac{3}{2}\right) = 1$$

Or compute the area of the triangle: $\frac{1}{2} \cdot 1 \cdot 2 = 1$

$F_X(x) = 0$ for $x < 0$,

$F_X(x) = 1$ for $x > 1$,

For $0 \leq x \leq \frac{1}{2}$: $F_X(x) = \int_0^x 4t\,dt = \left[2t^2\right]_0^x = 2x^2$

For $\frac{1}{2} \leq x \leq 1$: $F_X(x) = \int_0^{\frac{1}{2}} 4t\,dt \,!\!+ \int_{\frac{1}{2}}^{x} (4-4t)\,dt$! see next page for remark.

$$= \frac{1}{2} + \left[4t - 2t^2\right]_{\frac{1}{2}}^{x} = \frac{1}{2} + 4x - 2x^2 - \left(4 \cdot \tfrac{1}{2} - 2 \cdot \tfrac{1}{4}\right)$$

$$= 4x - 2x^2 - 1.$$! see next page for remark

(b) $E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx \overset{!}{=} \int_0^{\frac{1}{2}} x \cdot 4x\,dx + \int_{\frac{1}{2}}^{1} x(4-4x)\,dx$

$$= \left[\tfrac{4}{3}x^3\right]_0^{\frac{1}{2}} + \left[2x^2 - \tfrac{4}{3}x^3\right]_{\frac{1}{2}}^{1} = \frac{4}{3} \cdot \frac{1}{8} + \frac{2}{3} - \frac{1}{2} + \frac{1}{6} = \frac{1}{2}$$

Or $E(X) = \frac{1}{2}$ since $f_X(x)$ is symmetric in $x = \frac{1}{2}$.

(c) $P\left(\frac{X}{1-X} > 4\right) \overset{\text{since } 1-X \text{ non negative}}{=} P(X > 4 - 4X)$

$$= P\left(X > \tfrac{4}{5}\right).$$

(5c) continued

You can compute this in various ways.

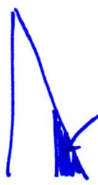
$$P(X > \tfrac{4}{5}) = \int_{\frac{4}{5}}^{1} (4 - 4x)\,dx$$

or $P(X > \frac{4}{5}) = P(X < \frac{1}{5})$ by symmetry

$$\left(= 2 \cdot \left(\tfrac{1}{5}\right)^2 = \tfrac{2}{25} \right)$$

or $P(X > \frac{4}{5}) = 1 - F_X(\frac{4}{5})$ and use b,

$$= 1 - \left(4 \cdot \tfrac{4}{5} - 2 \cdot \left(\tfrac{4}{5}\right)^2 - 1\right) = \tfrac{2}{25}.$$

or  Area triangle is $\frac{1}{2} \cdot \frac{1}{5} \cdot (4 - 4 \cdot \frac{4}{5}) = \frac{2}{25}$

For b: Observe that it makes no sense to say $E(X) = \cdots$ for $x < \frac{1}{2}$ and $\cdots$ for $x > \frac{1}{2}$, $E(x)$ is one real number, you should add the two integrals.

For a: If you compute $F_X(x)$ for $\frac{1}{2} < x \leq 1$, dont forget the part $\int_0^{\frac{1}{2}} 4t\,dt$!