Final Exam Probability Theory

July 3 2018, 8.45-11.30

- This exam consist of six exercises and a table. You can obtain 45 points. Your grade is given by (5+number of points)/5.
- You may use a simple calculator, but it is not allowed to use a graphical or a programmable calculator.
- Explain your answers clearly!

1. An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them and that they will pass the exam if they have at least 4 correct answers. Mary has figured out how to do 8 of the 10 problems.
(a) [3 points] Let X be the number of problems in the final exam that she cannot answer correctly. Compute the expectation of X.
(b) [2 points] Compute the conditional probability that she answered all questions of the exam correctly given that she passed the exam.

2. Suppose that the following information is known about twins: 30% of all twins that are born are 'identical', whereas the remaining 70% of all twins are 'fraternal'. Suppose further that half of all identical twins are both female, the other half are both male. Regarding fraternal twins 25% are known to be both female, 25% both male and the remaining 50% are male-female. A woman gives birth to twins.
(a) [2 points] Compute the probability that the twins are male-female.
(b) [3 points] Compute the conditional probability that the twins are fraternal if it is given that the twins are both male.

3. Let X and Y be independent random variables: X is uniformly distributed on the interval $[-1, 1]$ and Y is uniformly distributed on the interval $[0, 2]$.
(a) [3 points] Compute the density $f_Z(z)$ of the random variable $Z := X + Y$ for the situation that $z \in [1, 3]$.
(b) [3 points] Compute $P(Y > X^2)$.
(c) [3 points] Compute $\text{Cov}(X + 2Y, 3X - Y)$.

4. Let X and Y be continuous random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{6e^{xy}}{x^6} & \text{if } x > 1 \text{ and } y < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Show that the marginal density function of X is given by

$$f_X(x) = \begin{cases} \frac{6}{x^7} & \text{if } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) [3 points] Show that $E(Y|X=x) = -\frac{1}{x}$, for $x > 1$.
(c) [2 points] Compute $E(Y)$ using (a) and (b).

5. Let Y be a continous random variable with density function

$$f_Y(y) = \begin{cases} y+1 & \text{if } y \in [-1,0), \\ 1-y & \text{if } y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Compute the variance of Y.
(b) [3 points] Compute the density function of the random variable $V := |Y|$.
(c) [3 points] Compute the probability that the quadratic equation (in x)

$$x^2 - 2Yx - \frac{1}{2}Y = 0$$

has no real roots.

6. Let $X_1, X_2, X_3, \ldots, X_{100}$ be independent Poisson random variables with parameter 1 (and recall that both the expectation and the variance of a Poisson random variable are equal to its parameter).
(a) [3 points] Use the central limit theorem to give an approximation of $P(X_1 + X_2 + \cdots + X_{100} > 110)$.
(b) [3 points] Compute $P(X_1 + X_2 + X_3 = 2)$.
(c) [3 points] Let Y count how many of the random variables $X_1, X_2, \ldots, X_{100}$ are equal to zero. Give the probability mass function of Y, and give $E(Y)$ and $\text{Var}(Y)$.

Table

Area $\phi(x)$ under the standard normal curve to the left of x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998