Final Exam Probability Theory

June 1 2018, 15.15-17.15

- This exam consist of five exercises and a table. You can obtain 36 points. Your grade is given by (4+number of points)/4.
- You may use a simple calculator, but it is not allowed to use a graphical or a programmable calculator.
- Explain your answers clearly!

1. Let X and Y be continuous random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 6e^{2x}e^{5y} & \text{if } y \leq 0 \text{ and } x \leq -y, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Show that the marginal density function of Y is given by

$$f_Y(y) = \begin{cases} 3e^{3y} & \text{if } y \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) [3 points] Compute the marginal density function of X.
(c) [3 points] Show that $E(X|Y=y) = -y - \frac{1}{2}$, for $y \leq 0$.
(d) [3 points] Compute $E(X)$ using the results of (a) and (c).
(e) [4 points] Compute $P(Y < X)$.

2. Let X and Y be independent random variables and suppose that the density function of X is given by

$$f_X(x) = \begin{cases} \frac{4}{x^5} & \text{if } x > 1, \\ 0 & \text{otherwise,} \end{cases}$$

and that the density function of Y is given by

$$f_Y(y) = \begin{cases} 2y & \text{if } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [4 points] Compute the density function $f_Z(z)$ of $Z := X + Y$ for the the situation that $1 < z \leq 2$.
(b) [4 points] Compute $\text{Cov}(X + 2Y, X - Y)$.

3. [3 points] Let X be a continuous random variable with density function

$$f_X(x) = \begin{cases} |x| & \text{if } -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the density function of $Y := X^2$.

4. Suppose that the durations of phone calls in a call center (measured in minutes) are independent and exponentially distributed with parameter $\frac{1}{10}$. We denote the duration (in minutes) of the first 100 phone calls in this center today by $X_1, X_2, \ldots, X_{100}$. Recall that $E(X_i) = 10$ and that $\mathrm{Var}(X_i) = 100$.
(a) [3 points] Use the central limit theorem to give an approximation of

$$P\left(\frac{X_1 + X_2 + \cdots + X_{100}}{100} > 8.5\right).$$

(b) [3 points] We count how many of the first 100 phone calls today have a duration of less than 8 minutes and call this number Y. Give an approximation of $P(Y < 50)$ that is based on the central limit theorem. Don't forget the continuity correction!

5. [3 points] How many times would you expect to roll a fair die until all sides with an odd number have appeared at least once?

Table

Area ϕ(x) under the standard normal curve to the left of x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998