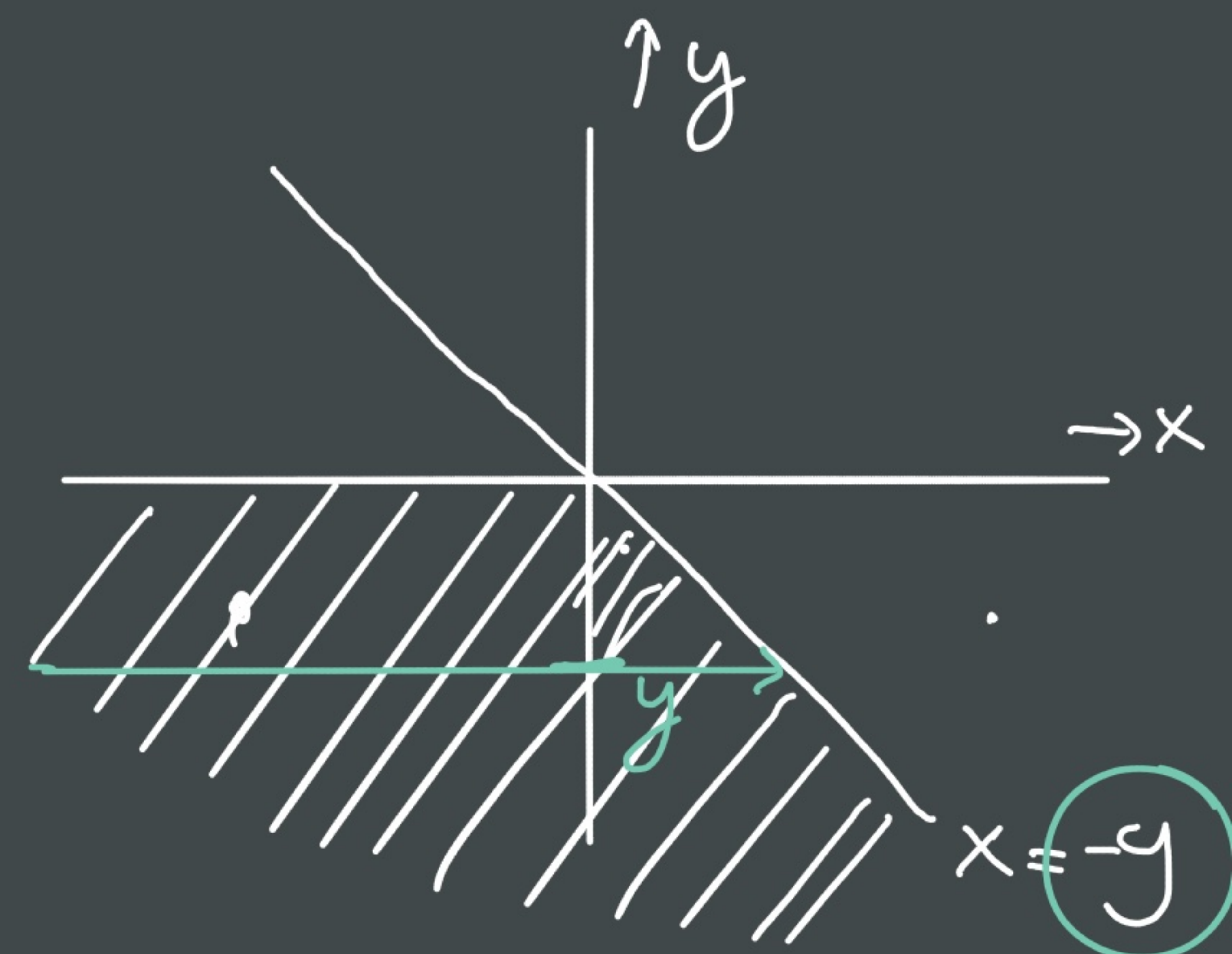
Final June 1

① $f_{X,Y}(x,y) = \begin{cases} 6e^{2x}e^{5y} & \text{if } y \le 0,\ x \le -y \\ 0 & \text{else} \end{cases}$




a) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

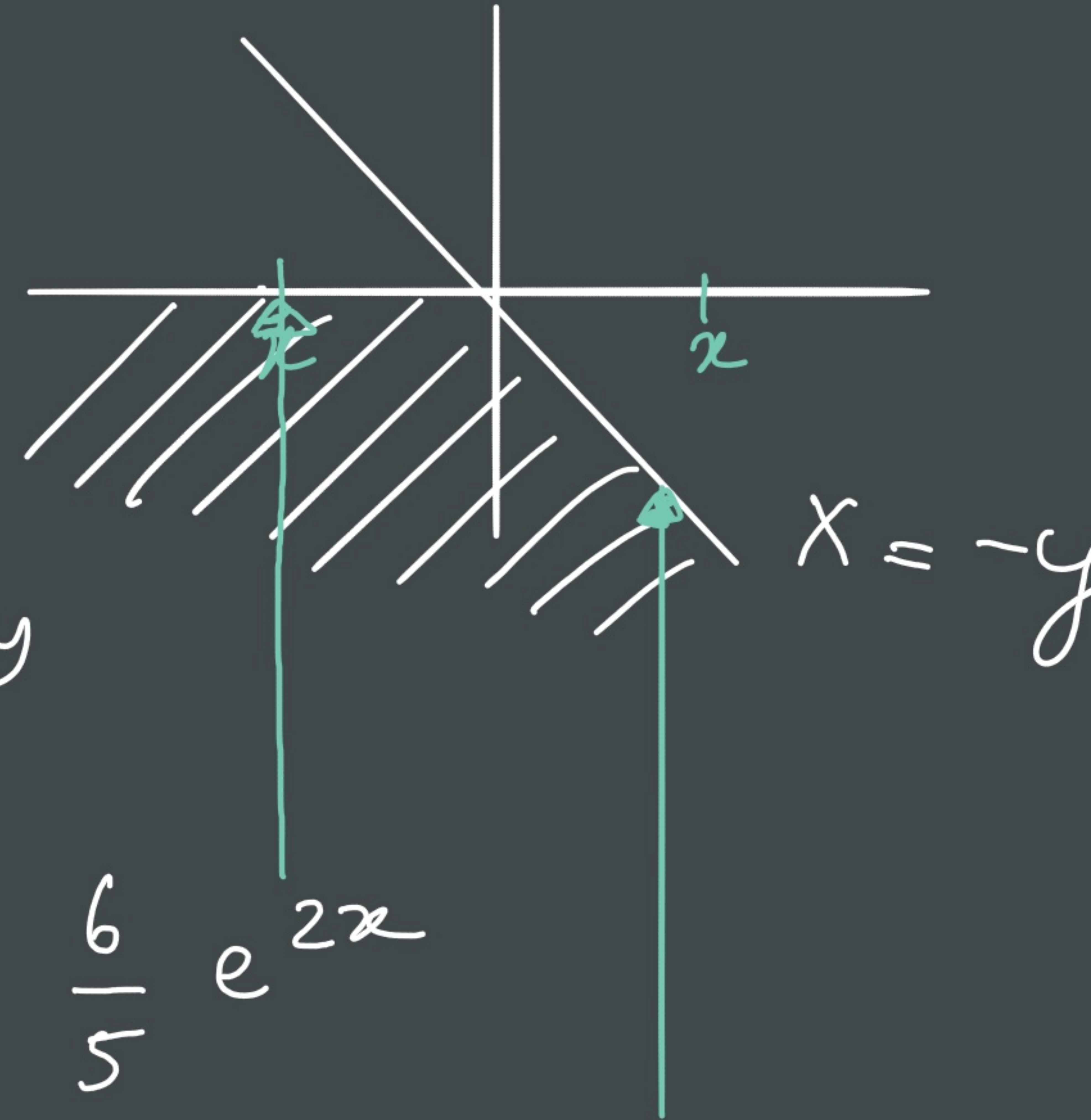
For $y>0$: $f_Y(y) = 0$

For $y \le 0$:

$$f_Y(y) = \int_{-\infty}^{-y} 6e^{2x}e^{5y}\,dx$$

$$= \left[3e^{2x}e^{5y}\right]_{-\infty}^{-y} = \underline{3e^{3y}}$$

b) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$




for $x<0$: $f_X(x) = \int_{-\infty}^{0} 6e^{2x}e^{5y}\,dy$

$$= \left[\frac{6}{5}e^{2x}e^{5y}\right]_{y=-\infty}^{0} = \frac{6}{5}e^{2x}$$

for $x \geq 0$: $f_X(x) = \int_{-\infty}^{-x} 6e^{2x}e^{5y}\,dy = \left[\frac{6}{5}e^{2x}e^{5y}\right]_{-\infty}^{-x}$

$$= \frac{6}{5}e^{-3x}.$$

(c) For $y \le 0$

$$E(X|Y=y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

$$= \int_{-\infty}^{-y} x \cdot \frac{6e^{2x}e^{5y}}{3e^{3y}}\,dx = \int_{-\infty}^{-y} x \cdot 2e^{2x}e^{2y}\,dx$$

$$= \Big[x \cdot e^{2x}e^{2y}\Big]_{-\infty}^{-y} - \int_{-\infty}^{-y} 1 \cdot e^{2x}e^{2y}\,dx$$

$$= -y - \Big[+\frac{1}{2}e^{2x}e^{2y}\Big]_{-\infty}^{-y} = \boxed{-y + \frac{1}{2}}$$

(d) $E(X) = \int_{-\infty}^{\infty} E(X \mid Y = y) \cdot f_Y(y)\,dy$

$$E(E(X \mid Y)) = E\left(-Y - \tfrac{1}{2}\right) = -E(Y) - \tfrac{1}{2}$$

$$= \int_{-\infty}^{0} \left(-y - \tfrac{1}{2}\right) \cdot 3e^{3y}\,dy$$

$$= \underbrace{\left[\left(-y - \tfrac{1}{2}\right) e^{3y}\right]_{y=-\infty}^{0} - \int_{-\infty}^{0} -e^{3y}\,dy}$$

$$= -\frac{1}{2} + \left[\frac{1}{3} e^{3y}\right]_{-\infty}^{0} = -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$$

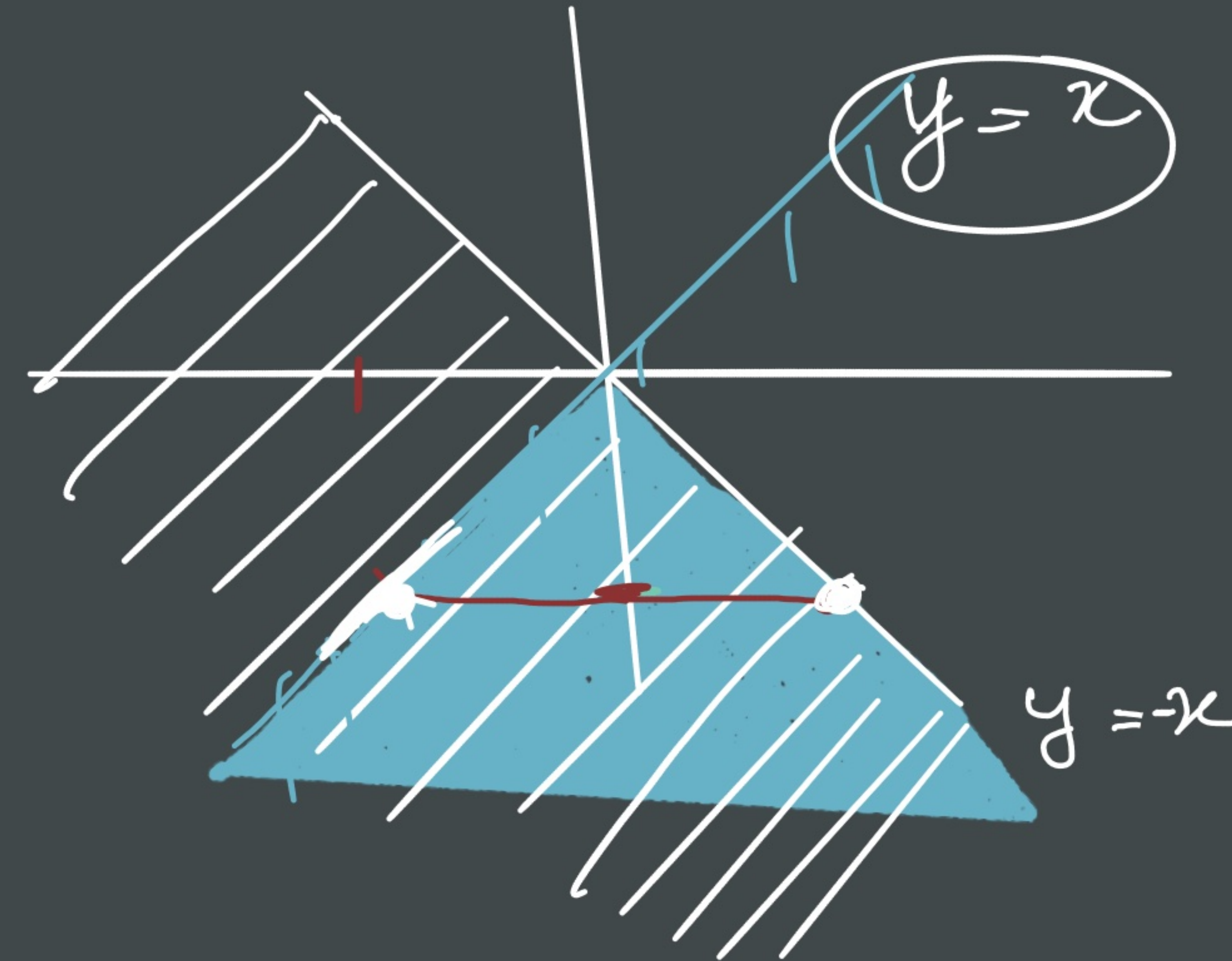
(e) $P(Y<X) =$

$$\int_{-\infty}^{0}\int_{y}^{-y} 6e^{2x}e^{5x}\,dx\,dy$$

$$= \int_{-\infty}^{0}\left[3e^{2x}e^{5x}\right]_{x=y}^{x=-y} dx$$

$$= \int_{-\infty}^{0}\left(3e^{+3y} - 3e^{7y}\right)dy = \left[+e^{3y} - \frac{3}{7}e^{7y}\right]_{-\infty}^{0}$$

$$= 1 - \frac{3}{7} = \frac{4}{7}.$$



② $X, Y$ independent

$$f_X(x) = \begin{cases} \frac{4}{x^5} & \text{if } x > 1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$z - x \geq 0$

$-x \geq -z$

$x \leq z$

$z - x \leq 1$

$-x \leq 1 - z$

$x \geq z - 1$

ⓐ $Z := X + Y$ Compute $f_Z(z)$ for $1 < z \leq 2$

$$f_Z(z) = \int_{-\infty}^{\infty} \underbrace{f_X(x)} f_Y(z-x)\,dx$$

$$= \int_1^{\infty} \frac{4}{x^5} \underbrace{f_Y(z-x)}\,dx = \int_1^{z} \frac{4}{x^5} \cdot \underbrace{2(z-x)}\,dx =$$

$$\int_1^z \frac{8z}{x^5} - \frac{8}{x^4}\,dx = \left[-\frac{2z}{x^4} + \frac{8}{3x^3}\right]_1^z$$

$$= -\frac{2}{z^3} + \frac{8}{3z^3} + 2z - \frac{8}{3}$$

$$= \frac{2}{3z^3} + 2z - \frac{8}{3}.$$

b) $\mathrm{Cov}(X+2Y, X-Y)$

$= \mathrm{Cov}(X,X) - \mathrm{Cov}(X,Y) + 2\,\mathrm{Cov}(X,Y) - 2\,\mathrm{Cov}(Y,Y)$

$= \mathrm{Var}(X) \underbrace{- \mathrm{Cov}(X,Y)}_{=0 \text{ by ind.}} + 2\,\mathrm{Var}(Y)$

$= \mathrm{Var}(X) - 2\,\mathrm{Var}(Y).$

- $E(X) = \int_1^\infty x \cdot \frac{4}{x^5}\,dx = \int_1^\infty \frac{4}{x^4}\,dx = \left[-\frac{4}{3} \cdot x^{-3}\right]_1^\infty = \frac{4}{3}$

$E(X^2) = \int_1^\infty x^2 \cdot \frac{4}{x^5}\,dx = \left[-2x^{-2}\right]_1^\infty = 2$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 2 - \frac{16}{9}$$

$$= \frac{2}{9}.$$

$$E(Y) = \int_0^1 2y^2 dx = \frac{2}{3} \quad ; \quad E(Y^2) = \int_0^1 2y^3 dy = \frac{1}{2}$$

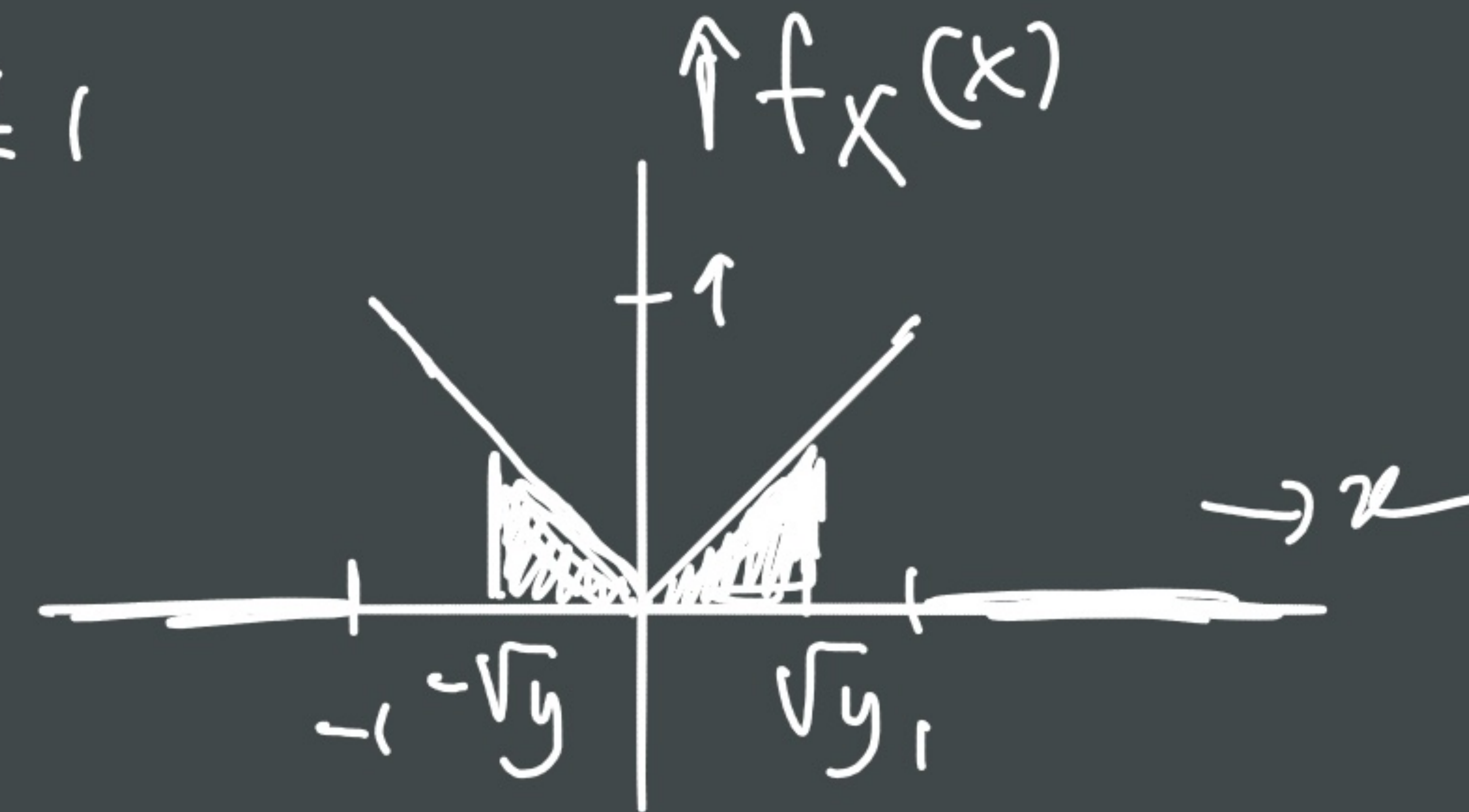
$$\mathrm{Var}(Y) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

$$\mathrm{Cov}(X+2Y, X-Y) = \frac{2}{9} - 2 \cdot \frac{1}{18} = \frac{1}{9}.$$

$$\operatorname{Cov}(X+2Y, X-Y) = E\big((X+2Y)(X-Y)\big) - E(X+2Y)E(X-Y)$$

③ (Final June 2018)

$$f_X(x) = \begin{cases} |x| & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$




$Y := X^2$ $\quad f_Y(y)$ ?

$F_Y(y) = P(X^2 \leq y) = 0$ for $y < 0$

$F_Y(y) = 1$ for $y > 1$.

For $y \in [0,1]$

$$F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{0} -x \, dx + \int_{0}^{\sqrt{y}} x \, dx = y$$

So $f_Y(y) = 1$ for $y \in [0,1]$

$f_Y(y) = 0$ otherwise

($Y$ is uniform on $[0,1]$).

4 a

$$P\left(\frac{X_1+\cdots+X_{100}}{100} > 8{,}5\right)$$

$$\frac{X_1+\cdots+X_n-n\mu}{\sqrt{n}\cdot\sigma}$$

$$= P\left(X_1+\cdots+X_{100} > 850\right)$$

$$= P\left(\frac{X_1+\cdots+X_{100}-100\cdot 10}{\underset{\sqrt{n}}{\sqrt{100}}\ \underset{\sigma}{\sqrt{100}}} > \underbrace{\frac{850-100\cdot 10}{10\cdot 10}}_{-1{,}5}\right)$$

$$\approx 1-\Phi(-1{,}5) = 1-\left(1-\Phi(1{,}5)\right) = \Phi(1{,}5) =$$

$$0{,}9332.$$

(b) $Y \sim bin(n=100, \; p = P(X_i < 8))$

$$P(X_i < 8) = \int_0^8 \frac{1}{10} e^{-\frac{1}{10}x} \, dx$$

$Y$ approx $N(np, \sqrt{np(1-p)})$

$$= \left[ -e^{-\frac{1}{10}x} \right]_0^8 = -e^{-0,8} + 1 \approx 0,551.$$

$Y \sim bin(n=100, p=0,551)$

$$P(Y < 50) = P(Y \leq 49\tfrac{1}{2}) \overset{CLT}{\approx} \Phi\left( \frac{49\frac{1}{2} - 100 \cdot 0,551}{\sqrt{100 \cdot 0,551 \cdot 0,449}} \right)$$

$$\approx \Phi(-1,13) = 1 - \Phi(1,13) = 1 - 0,8708$$

(5) Let $X_1$ be # rolls until the first odd outcome.

$X_2$ the number of rolls after $X_1$ until the next 'new' odd outcome

$X_3$ the number of additional rolls until the last odd number I am waiting for.

$$E(X_1+X_2+X_3) = E(X_1)+E(X_2)+E(X_3)$$

$$X_1 \sim \text{geom}\left(p=\tfrac{1}{2}\right) \rightarrow E(X_1)=2$$

$$X_2 \sim \text{geom}\left(p=\tfrac{1}{3}\right) \rightarrow E(X_2)=3$$

$$X_3 \sim \text{geom}\left(p=\tfrac{1}{6}\right) \rightarrow E(X_3)=6$$

$$E(X_1+X_2+X_3) = 2+3+6 = 11.$$