16/3/2020.

Answers Midterm exam March 2018

Advice:

First do the complete exam without checking the answers.

I am feeling old and using a preachy tone but really: if you just read answers, they usually look quite obvious and logical (it's mathematics ☺)

So:
- first try the problems yourself
- check if you did it correctly
- if not: have a look at the theory again to help you understand better.
- try to do the problems that went wrong again.

After you dealt with this exam in the above manner, try the midterm of March 2019 later this week and see if you can handle this maybe even better than 2018's version.

Wish you all the best, hope you are all doing well and not feeling ill. It are difficult times, take care!

Best, Carni Quant

disclaimer: errors in writing or computations possible, apologies....

Midterm March 2018

Problem 1

Choose 2 socks out of n socks, without replacement.
You can do this in 2 ways:

1. Don't take the order into account.
   The sample space consists of $\binom{n}{2}$ outcomes
2. Take the order in which the socks are taken into account, then there are $n \cdot (n-1)$ outcomes.

a) Method 1: $P(\text{both socks red}) = \dfrac{\binom{3}{2}}{\binom{n}{2}} = \dfrac{3}{\frac{n(n-1)}{2}} = \dfrac{6}{n(n-1)}$

Method 2: $P(\text{both socks red}) = \dfrac{3 \cdot 2}{n(n-1)} = \dfrac{6}{n(n-1)}$

Or: $P(\text{both socks red}) = P(\text{first sock red}) \cdot P(\text{second sock red} \mid \text{first sock red}) = \frac{3}{n} \cdot \frac{2}{n-1}$ (Method 3 in fact..)

Then: $\dfrac{6}{n(n-1)} = \dfrac{1}{2}$ gives

$n(n-1) = 12$, so
$n^2 - n - 12 = (n-4)(n+3) = 0$,
So $n = 4$ (since n cannot be negative)

b) We can see one colour (two black socks or two red socks) or two colours.

$P(X=1) = P(\text{both red}) + P(\text{both black})$

$$= \left\{ \begin{array}{ll} \dfrac{3 \cdot 2}{7 \cdot 6} + \dfrac{4 \cdot 3}{7 \cdot 6} & \text{(method 2)} \\[2ex] \dfrac{\binom{3}{2}}{\binom{7}{2}} + \dfrac{\binom{4}{2}}{\binom{7}{2}} & \text{(method 1)} \\[2ex] \dfrac{3}{7} \cdot \dfrac{2}{6} + \dfrac{4}{7} \cdot \dfrac{3}{6} & \text{(method 3)} \end{array} \right\} = \frac{18}{42} = \frac{3}{7}.$$

1b) Continued:

$P(X=2) = 1 - P(X=1) = 1 - \frac{3}{7} = \frac{4}{7}$.

(or $P(X=2) = \frac{\binom{3}{1}\binom{4}{1}}{\binom{7}{2}} = \frac{21}{21} = \frac{4}{7}$, method 1

or $P(X=2) = P(\text{first red, second black}) + P(\text{first black, second red})$

$= \frac{3}{7} \cdot \frac{4}{6} \cdot 2$.

$Var(X) = E(X^2) - (E(X))^2$

By def: $Var(X) = E(X - E(X))^2$ but we showed that this is equal to $E(X^2) - (E(X))^2$ which is often easier in computations

$E(X) = \sum_x x P(X=x) = 1 \cdot \frac{3}{7} + 2 \cdot \frac{4}{7} = \frac{11}{7}$

$E(X^2) = \sum_x x^2 P(X=x) = 1^2 \cdot \frac{3}{7} + 2^2 \cdot \frac{4}{7} = \frac{19}{7}$.

$Var(X) = \frac{19}{7} - \left(\frac{11}{7}\right)^2 = \frac{133-121}{49} = \frac{12}{49}$.

c): A: both have same colour
B: at least one red

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

Just use the definition of conditional probability here

$A \cap B$ is the event that both socks are red,
so $P(A \cap B) = \frac{3}{7} \cdot \frac{2}{6} = \frac{1}{7}$ (or $\frac{\binom{3}{2}}{\binom{7}{2}}$ or ...)

$P(B) = 1 - P(B^c) = 1 - P(\text{both black})$

$= 1 - \frac{4}{7} \cdot \frac{3}{6} = \frac{5}{7}$

So $P(A|B) = \frac{\frac{1}{7}}{\frac{5}{7}} = \frac{1}{5}$

Problem 2 Write T_1 for the event that the first inhabitant spoke truthfully and S_2 for the event that the second says that the first spoke truthfully

$$P(T_1 \mid S_2) \left[= \frac{P(T_1 \cap S_2)}{P(S_2)}\right]$$

$$= \frac{P(T_1)P(S_2 \mid T_1)}{P(T_1)P(S_2 \mid T_1) + P(T_1^c)P(S_2 \mid T_1^c)}$$

by Bayes

This is typically something that can be computed with Bayes' formula, since I ask you to compute $P(T_1|S_2)$, but $P(S_2|T_1)$ and $P(S_2|T_1^c)$ are known.

$$= \frac{\frac{1}{3}\cdot\frac{1}{3}}{\frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3}}$$

If T_1^c, then the first lies. Given that, the second one must lie if he says that person 1 spoke thruthfully. So $P(S_2|T_1^c) = \frac{2}{3}$.

$$= \frac{\frac{1}{9}}{\frac{5}{9}} = \frac{1}{5}.$$

Problem 3

Write: G: defendant is guilty
X: number of jurors that vote guilty
C: defendant is convicted

a) $P(\text{correct decision}) = P(G \cap C) + P(G^c \cap C^c)$

since decision is correct if an innocent defendent is not convicted $(G^c \cap C^c)$ or a guilty defendent is convicted $(G \cap C)$.

$$= P(G)P(C|G) + P(G^c)P(C^c|G^c)$$

$$= P(G)\,P(X=10 \text{ or } X=11 \text{ or } X=12 \mid G) + P(G^c)\left(1 - P(X=10 \text{ or } X=11 \text{ or } X=12 \mid G^c\right)$$

$=$

I use binomial probabilities Conditioned on G the 'success' prob. is $0{,}8$; conditioned on G^c the success prob. is $0{,}1$ if 'success' corresponds to the prob. that a juror votes 'guilty'

$$= 0{,}65\left(\binom{12}{10}(0{,}8)^{10}\cdot(0{,}2)^2 + \binom{12}{11}\cdot(0{,}8)^{11}\cdot(0{,}2) + (0{,}8)^{12}\right) +$$
$$0{,}35\left(1 - \left\{\binom{12}{10}(0{,}1)^{10}\cdot(0{,}9)^2 + \binom{12}{11}\cdot(0{,}1)^{11}\cdot 0{,}9 + (0{,}1)^{12}\right\}\right)$$

$$\approx 0{,}65 \cdot 0{,}7721 + 0{,}35\left(1 - 5{,}455\cdot 10^{-9}\right) \approx 0{,}85.$$

b) $P(\text{Conviction}) = P(G \cap C) + P(G^c \cap C)$

$$= P(G)\,P(X=10 \text{ or } X=11 \text{ or } X=12 \mid G) + P(G^c)\,P(X=10 \text{ or } X=11 \text{ or } X=12 \mid G^c)$$

$$= \underbrace{0{,}65 \cdot 0{,}7721}_{\text{we did this in a}} + 0{,}35\,\underbrace{\left(\binom{12}{10}(0{,}1)^{10}\cdot 0{,}9^2 + \binom{12}{11}(0{,}1)^{11}\cdot 0{,}9 + (0{,}1)^{12}\right)}$$

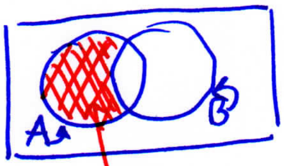
this is practically zero so we don't see the contribution of this term in the final answer. The prob. that an innocent defendent is convicted is (luckily) ≈ 0.

$\approx 0{,}502$

Problem 4

Given: $P(A^c) = 0{,}3$ $P(B^c) = 0{,}4$ and $P(A \cup B) = 0{,}88$.

(a) A occurs but B not means $A \cap B^c$.



$$P(A \cap B^c) = P(A \cup B) - P(B)$$
$$= 0{,}88 - (1 - P(B^c))$$
$$= 0{,}88 - (1 - 0{,}4) = 0{,}28.$$

Alternative solution:

$P(A \cap B^c) = P(A) - P(A \cap B)$. (*)

$P(A) = 1 - P(A^c) = 1 - 0{,}3 = 0{,}7$; $P(B) = 1 - P(B^c) = 0{,}6$.

From $P(A \cup B) = 0{,}88$ we conclude:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$,

So $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0{,}7 + 0{,}6 - 0{,}88 = 0{,}42$.

Insert this in (*) to find:

$P(A \cap B^c) = 0{,}7 - 0{,}42 = 0{,}28$

b) We check whether $P(A \cap B^c) = P(A) \cdot P(B^c)$.

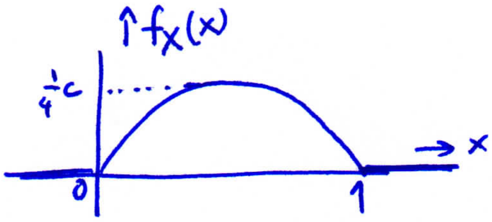
Since $0{,}28 = 0{,}7 \cdot 0{,}4$ this is indeed true.

So A and B^c are independent and therefore also A and B are independent.

You can also check that $P(A \cap B) = P(A)\,P(B)$ here, depending on what you did in (a).

See def 2.17 and fact 2.20.

Problem 5



a) $\int_0^1 cx(1-x)\,dx = \left[\frac{1}{2}cx^2 - \frac{1}{3}cx^3\right]_0^1 = \frac{1}{2}c - \frac{1}{3}c = \frac{1}{6}c$

So $\int_{-\infty}^{\infty} f_X(x) = 1$ gives that $\frac{1}{6}c = 1$, so $c = 6$.

b) Recall that $F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$, since X is a continuous r.v.

So for $x<0$: $F_X(x) = 0$

For $x \in [0,1]$: $F_X(x) = \int_0^x 6t(1-t)\,dt = \int_0^x (6t - t^2)\,dt$

$= \left[3t^2 - 2t^3\right]_0^x = 3x^2 - 2x^3$.

For $x > 1$: $F_X(x) = 1$.

c) $Var(3X-7) = 9\,Var(X)$ .... Since $Var(aX+b) = a^2\,Var(X)$

$Var(X) = E(X^2) - (E(X))^2$.

$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^1 x(6x - 6x^2)\,dx$

$= \left[2x^3 - \frac{3}{2}x^4\right]_0^1 = \frac{1}{2}$.

$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx = \int_0^1 x^2(6x - 6x^2)\,dx$

$= \left[\frac{6}{4}x^4 - \frac{6}{5}x^5\right]_0^1 = \frac{3}{10}$

So $Var(X) = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$, and

$Var(3X-7) = 9 \cdot \frac{1}{20} = \frac{9}{20}$.

⑥ $X \sim N(\mu = 23{,}25,\ \sigma^2 = (0{,}1)^2)$, where X is the diameter of a 1 euro coin.

$\boxed{a)}$ $P(\text{coin is not accepted}) = P(X < 22{,}95 \text{ or } X > 23{,}45)$

$= P(X < 22{,}95) + P(X > 23{,}45)$

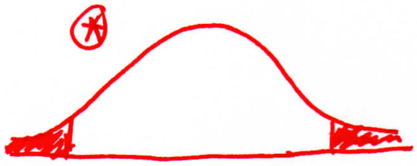
$$= P\left(\frac{X - 23{,}25}{0{,}1} < \underbrace{\frac{22{,}95 - 23{,}25}{0{,}1}}_{-3}\right) + P\left(\frac{X - 23{,}25}{0{,}1} > \underbrace{\frac{23{,}45 - 23{,}25}{0{,}1}}_{2}\right)$$

Since if $X \sim N(\mu, \sigma^2)$ then $\frac{X-\mu}{\sigma} \sim N(0,1)$

So we standardise in order to be able to use the table

$= \Phi(-3) + (1 - \Phi(2))$

$= (1 - \Phi(3))^{*} + (1 - \Phi(2))$



* $\Phi(-z) = 1 - \Phi(z)$.

$= 2 - \Phi(3) - \Phi(2) = 2 - 0{,}9987 - 0{,}9772 = 0{,}0241$

Other option $P(\text{coin is not accepted}) = 1 - P(22{,}95 \le X \le 23{,}25)$, then standardise.

$\boxed{b)}$ $P(|X - 23{,}25| < d) = P(-d < X - 23{,}25 < d) =$

$P\left(\frac{-d}{0{,}1} < \frac{X - 23{,}5}{0{,}1} < \frac{d}{0{,}1}\right) = \Phi\left(\frac{d}{0{,}1}\right) - \Phi\left(\frac{-d}{0{,}1}\right)$

$= \Phi\left(\frac{d}{0{,}1}\right) - \left(1 - \Phi\left(\frac{d}{0{,}1}\right)\right)^{*} = 2\Phi\left(\frac{d}{0{,}1}\right) - 1.$

this must be approximately $0{,}95$, so $\Phi\left(\frac{d}{0{,}1}\right) = 0{,}975$. Look in the table and find that $\Phi(1{,}96) = 0{,}9750$.

So $\frac{d}{0{,}1} \approx 1{,}96$ and $d \approx 0{,}196$.

Remember! $\Phi(x) = P(Z \le x)$ if $Z \sim N(0,1)$.