Final Exam Probability Theory

July 4 2017, 8.45-11.30

- This exam consist of six exercises and a table. You can obtain 45 points. Your grade is given by (5+number of points)/5.
- You may use a simple calculator, but it is not allowed to use a graphical or a programmable calculator.
- Explain your answers clearly!

1. Urn A contains 4 red balls and 1 white ball and urn B contains 1 red ball and 4 white balls. A random ball from urn A is chosen and placed in urn B. After that, 2 random balls from urn B are chosen (at the same moment, without replacement). Let R be the event that a red ball is chosen from urn A and let X be the number of red balls that are obtained from urn B.
(a) [3 points] Compute $P(R|X = 1)$.
(b) [3 points] Compute the probability mass function of X and use this to compute the variance of X.

2. You roll a fair dice 4 times. Let X be the highest number that appears, let Y be the number of times that an even number appears and let Z be the sum of the four outcomes.
(a) [3 points] Compute the cumulative distribution function of X.
(b) [3 points] Give the probability mass function of Y and give $E(Y)$ and $\text{Var}(Y)$.
(c) [3 points] Compute $P(Z > 5)$.
(d) [3 points] Compute the probability that the outcome of Z is even (hint: for which values of Y is this the case?)

3. Let X and Y be continuous random variables with joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{4}{x^6} & \text{if } x > 1 \text{ and } 0 < y < x, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Compute the marginal density function of Y.
(b) [2 points] Show that the marginal density function of X is given by

$$f_X(x) = \begin{cases} \frac{4}{x^5} & \text{if } x > 1, \\ 0 & \text{otherwise,} \end{cases}$$

(c) [2 points] Show that $E(Y|X = x) = \frac{1}{2}x$, for $x > 1$.
(d) [2 points] Compute $E(Y)$ using (b) and (c).
(e) [3 points] Compute the density of $Z := -\ln(X)$.

4. Let X and Y be independent random variables and suppose that the density function of X is given by

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that the density function of Y is given by

$$f_Y(y) = \begin{cases} 3e^{-3y} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Compute the density function of $Z := X + Y$.
(b) [3 points] Compute $P(Y < 2X)$.

5. Let $X_1, \ldots, X_{100}$ be independent and identically distributed continuous random variables with $E(X_i) = 5$ and $\text{Var}(X_i) = 25$.
(a) [3 points] Use the Central Limit Theorem to give an approximation of

$$P(X_1 + X_2 + \cdots + X_{100} > 575).$$

(b) [3 points] Compute $\text{Cov}(X_1 + X_2 - X_3, X_2 + X_3 + X_4)$.

6. [3 points] I let my computer generate independent outcomes of uniform random variables on the interval [0,1] and keep on doing this until I have obtained an outcome that is greater than 0.75 exactly four times. Let X be the number of random variables that have been generated at that moment. Give the probability mass function and the expectation of X.