Final Exam Probability Theory

June 2 2017, 15.15-17.15

- This exam consist of five exercises and a table. You can obtain 36 points. Your grade is given by (4+number of points)/4.
- You may use a simple calculator, but it is not allowed to use a graphical or a programmable calculator.
- Explain your answers clearly!

1. Let X and Y be continuous random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 9x^2y^2 & \text{if } 0 < y < 1 \text{ and } -y \leq x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Compute the marginal density function of X.
(b) [3 points] Show that the marginal density function of Y is given by

$$f_Y(y) = \begin{cases} 6y^5 & \text{if } 0 < y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

(c) [3 points] Show that $E(X^2|Y = y) = \frac{3}{5}y^2$, for $0 < y < 1$.
(d) [3 points] Compute $E(X^2)$ using (b) and (c).
(e) [3 points] Compute the covariance between X and Y.

2. Let X and Y be independent random variables and suppose that the density function of X is given by

$$f_X(x) = \begin{cases} e^{-x+1} & \text{if } x > 1, \\ 0 & \text{otherwise,} \end{cases}$$

and that the density function of Y is given by

$$f_Y(y) = \begin{cases} ye^{-y} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Compute the density function of $Z := X + Y$.
(b) [3 points] Compute $P(Y > X)$.

3. Let X be an exponentially distributed random variable with parameter $\frac{1}{5}$.
(a) [3 points] Explain what is meant by the statement that X is memoryless.
(b) [3 points] Compute the density function of $Y := e^{-\frac{X}{5}}$

4. [3 points] Ten Business Analytics students and six Mathematics students go for dinner and are randomly seated at a round table. Let X be the number of Business Analytics students for whom one neighbour is a Business Analytics student and the other a Mathematics student. Compute $E(X)$.

5. Let $X_1, \ldots, X_{100}$ be independent standard normal random variables.
(a) [3 points] Compute

$$P\left(-1 \leq \frac{X_1 + X_2 + X_3 + X_4}{4} \leq 1\right).$$

(b) [3 points] Let Y be equal to the number of indices $i \in \{1, 2, \ldots, 100\}$ for which $X_i > 0$. Give an approximation of $P(45 < Y \leq 52)$ that is based on the Central Limit Theorem, don't forget the continuity correction!

Table

Area ϕ(x) under the standard normal curve to the left of x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998