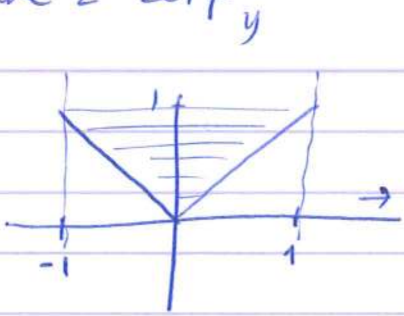
Final Exam Probability theory June 2 2017



1 a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

For $x \notin (-1,1)$: $f_X(x) = 0$

For $x \in (-1,0)$ $f_X(x) = \int_{-x}^{1} 9x^2y^2\, dy = \left[3x^2y^3\right]_{-x}^{1} = 3x^2 + 3x^5$

For $x \in [0,1)$ $f_X(x) = \int_{x}^{1} 9x^2y^2\, dy = \left[3x^2y^3\right]_{x}^{1} = 3x^2 - 3x^5.$

b) For $0 < y < 1$ $f_Y(y) = \int_{-y}^{y} 9x^2y^2\, dx = \left[3x^3y^2\right]_{-y}^{y} = 3y^5 + 3y^5 = 6y^5$

For $y \notin (0,1)$ $f_Y(y) = 0$

c) $E(X^2 | Y=y) = \int_{-y}^{y} x^2 \cdot f_{X|Y}(x|y)\, dx = \int_{-y}^{y} x^2 \cdot \frac{f_{X,Y}(x,y)}{f_Y(y)}\, dx$

$= \int_{-y}^{y} x^2 \cdot \frac{9x^2y^2}{6y^5}\, dx = \left[\cancel{9}\frac{3}{2} \cdot \frac{1}{5} \cancel{9}\frac{x^5}{y^3}\right]_{x=-y}^{x=y}$

$= \frac{3}{5} y^2.$

d) $E(X^2) = \int_0^1 \underbrace{\frac{3}{5}y^2}_{E(X^2|Y=y)} \cdot \underbrace{6y^5}_{f_Y(y)}\, dy = \left[\frac{18}{5} \cdot \frac{1}{8} y^8\right]_0^1 = \frac{18}{40} = \frac{9}{20}$

e) $Cov(X,Y) = E(XY) - E(X)E(Y)$

$E(XY)$ and $E(X)$ are both zero by symmetry, so $Cov(X,Y) = 0$

Or compute: $E(XY) = \int_0^1 \int_{-y}^{y} xy \cdot 9x^2y^2 \, dx \, dy = \int_0^1 \left[\frac{9}{4}x^4y^3\right]_{-y}^{y} dy$

$= 0$

and similarly $E(X) = 0$.

2a) $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, dx$

$= \int_1^{\infty} e^{-x+1} f_Y(z-x) \, dx.$

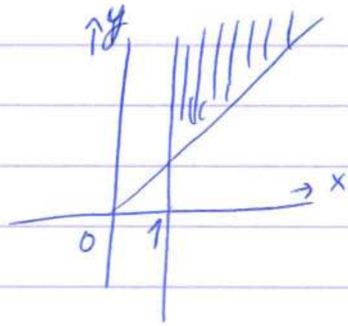
So $f_Z(z) = 0$ if $z \leq 1$.

For $z > 1$, $z - x \geq 0$ gives $x \leq z$.

So for $z > 1$ $f_Z(z) = \int_1^z e^{-x+1} \cdot (z-x) e^{-(z-x)} \, dx$

$= \int_1^z (z-x) e^{-z+1} \, dx = \left[zx - \frac{x^2}{2}\right]_1^z \cdot e^{-z+1}$

$= \left(\frac{1}{2}z^2 - z + \frac{1}{2}\right) e^{-z+1}.$



b) $P(Y > X) = \int_1^{\infty} \int_x^{\infty} f_X(x) f_Y(y) \, dy \, dx$

$= \int_1^{\infty} e^{-x+1} \left[-ye^{-y} - e^{-y}\right]_x^{\infty} dx = \int_1^{\infty} e^{-x+1} (xe^{-x} + e^{-x}) \, dx$

$= \int_1^{\infty} (xe^{-2x+1} + e^{-2x+1}) \, dx = \left[x \cdot \left(-\frac{1}{2}e^{-2x+1}\right)\right]_1^{\infty} - \int_1^{\infty} -\frac{1}{2}e^{-2x+1} + \int_1^{\infty} e^{-2x+1}$

$= \frac{1}{2}e^{-1} + \left[\frac{3}{2} \cdot \left(-\frac{1}{2}\right) e^{-2x+1}\right]_1^{\infty} = \frac{1}{2}e^{-1} + \frac{3}{4}e^{-1} = \frac{5}{4}e^{-1}.$

3a) That for $s,t > 0$ $P(X > s+t \mid X > s) = P(X > t)$.

or, that, if we interpret X as a 'life time' or 'waiting time' conditionally on the event that $X > s$, the remaining waiting time/life time after s has the same distribution as the original waiting/life time.

b) $$f_X(x) = \begin{cases} \frac{1}{5} e^{-\frac{1}{5}x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$Y = e^{-X/5}$ so $\ln Y = -\frac{X}{5}$, so $X = -5 \ln Y$.

$$\text{So } f_Y(y) = \begin{cases} f_X(-5\ln y) \cdot \left|\frac{-5}{y}\right| & \text{if } 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{5} e^{-\frac{1}{5} \cdot -5 \ln y} \cdot +\frac{5}{y} = \frac{1}{5} \cdot y \cdot \frac{5}{y} = 1 & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(so Y is uniform on $(0,1]$).

4) Number the chairs from 1 to 16.

Let $X_i = \begin{cases} 1 & \text{If at chair } i \text{ a BA student is seated with a math student at one side and a BA student at the other} \\ 0 & \text{otherwise} \end{cases}$

$X = X_1 + \cdots + X_{16}$, so $E(X) = E(X_1) + \cdots + E(X_{16})$

$E(X_i) = \frac{10}{16} \cdot \frac{9}{15} \cdot \frac{6}{14} \cdot 2.$

$E(X) = 16 \cdot E(X_1) = \frac{10 \cdot 9 \cdot 6 \cdot 2}{15 \cdot 14}$

5a) $X_1 + X_2 + X_3 + X_4 \sim N(\mu = 0,\ \sigma^2 = 4)$

$$P\left(-1 \leq \frac{X_1 + \cdots + X_4}{4} \leq 1\right) = P(-4 \leq X_1 + \cdots + X_4 \leq 4)$$

$$= P\left(-2 \leq \frac{X_1 + \cdots + X_4}{2} \leq 2\right) = \Phi(2) - \Phi(-2)$$

$$= \Phi(2) - (1 - \Phi(2)) = 2\Phi(2) - 1 = 2 \cdot 0{,}9772 - 1$$

$$= 0{,}9544.$$

5b) $Y \sim \text{bin}(n = 100,\ p = \frac{1}{2})$

$$P(45 < Y \leq 52) = P(45\tfrac{1}{2} \leq Y \leq 52\tfrac{1}{2})$$

$$= P\left(\frac{45\frac{1}{2} - \frac{1}{2} \cdot 100}{\sqrt{\frac{1}{2} \cdot \frac{1}{2} \cdot 100}} \leq \frac{Y - 50}{5} \leq \frac{52\frac{1}{2} - 50}{5}\right)$$

$$\approx \Phi(0{,}5) - \Phi(-0{,}90) = 0{,}6915 - (1 - 0{,}8159)$$