# Midterm Probability Theory

March 30, 2017, 12.00-14.00

- This midterm consists of five exercises and a table. You can obtain 36 points. Your grade is given by (4+number of points)/4.
- You may use a simple calculator, but it is not allowed to use a graphical or a programmable calculator.
- Explain your answers clearly.

**1.** An urn contains 12 marbles of which 3 are red, 3 yellow, 3 green and 3 blue. We select 3 arbitrary marbles from this urn without replacement (and without order, so we take the 3 marbles at the same moment). Let $A$ be the event that at least one of the marbles is yellow and $B$ be the event that at least one of the marbles is red.
(a) [2 points] Show that $P(A) = P(B) = \frac{34}{55}$.
(b) [2 points] Show that $P(A \cup B) = \frac{10}{11}$ by first computing the probability of the complement of $A \cup B$.
(c) [3 points] Compute $P(A \cap B)$ using the answers of (a) and (b).

We put the marbles back in the urn. We now select arbitrary marbles one by one from the urn and without replacement, until we have obtained 2 red marbles in total. Let $X$ be the number of marbles we take and let $C$ be the event that the second marble selected is red.
(d) [3 points] Compute $P(X = 5)$.
(e) [3 points] Compute $P(X = 3 \mid C)$.

**2.** Suppose an aircraft engine will fail in flight with probability $0 < p < 1$ independently of the plane's other engines. Also suppose that a plane can complete the journey successfully if at least half of its engines do not fail.
(a) [3 points] Consider a plane with five engines. Let $X$ be the number of engines that fail during the next flight. Give the probability mass function, the expectation, the variance and the name of the distribution of $X$.
(b) [3 points] For what values of $p$ is a four-engine plane preferable to a two-engine plane?

**3** [3 points] We repeatedly toss a fair coin until we see 'heads'. Let $X$ be the number of tosses we need. Compute the probability that $X$ is a multiple of 3.

**Please turn over!**

**4.** Let $X$ be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{2}{x^2} & \text{if } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Compute $P(1 < X < \frac{5}{4} \mid X < \frac{3}{2})$
(b) [3 points] Compute $\text{Var}(X)$.
(c) [3 points] Let $Y := \frac{1}{X^2}$. Compute $E(Y)$.

**5.** For this exercise you need the table. Suppose the annual rainfall in Cleveland, Ohio, is normally distributed with mean 40.2 inches and standard deviation 8.4 inches.
(a) [2 points] Compute the probability that next years rainfall will exceed 44.0 inches.
(b) [3 points] Let $Y$ be the number of years it takes until we have observed an annual rainfall of less than 38.0 inches. Compute $E(Y)$. Assume that if $A_i$ is the event that the rainfall is less than 38.0 inches in year $i$ (from now), then the events $A_i, i \geq 1$ are independent.

**Table**

Area ϕ(x) under the standard normal curve to the left of x

| x | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| .1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| .2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| .3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| .4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| .5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| .6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| .7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| .8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| .9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |