Resit Probability Theory

June 28, 2016, 8.45-11.30

- This exam consists of 6 exercises and a table. You can obtain 45 points. Your grade is given by (5+number of points)/5.
- You may use a simple calculator, but it is not allowed to use a graphical or a programmable calculator.
- Explain your answers clearly.

1. An urn contains 4 white and 2 black balls. A fair dice is rolled and that number of balls is randomly chosen from the urn (without replacement).
(a) [3 points] Compute the probability that at least one of the selected balls is white.
(b) [3 points] Compute the conditional probability that the dice landed on 3 given that all the balls selected are white.
(c) [3 points] The balls are placed back in the urn. We randomly select 4 balls from the urn, **with** replacement. Let A be the event that the first ball is white, and B the event that there is at least one black ball in the selection. Are A and B independent?

2. [3 points] Let X have a Poisson distribution with expectation 2.2. Compute $P(X = 2 \mid X \geq 1)$.

3. [3 points] Let X be a normally distributed random variable with expectation 3.1 and suppose that $P(X > 4.1) = 0.15$. Compute $P(X < 2.5)$.

4. Let X and Y be continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{y^3} e^{-y(x-y)} & \text{if } y > 1 \text{ and } x > y, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Show that the marginal density function of Y is given by

$$f_Y(y) = \begin{cases} \frac{3}{y^4} & \text{if } y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) [3 points] Show that $E(X \mid Y = y) = y + \frac{1}{y}$ for $y > 1$.
(c) [2 points] Compute $E(X)$.
(d) [3 points] Compute $\mathrm{Var}(Y)$.
(e) [3 points] Compute the density function of $Z := \ln(Y)$

5. Let X and Y be independent random variables with density functions

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{y}{2} & \text{if } 0 \leq y \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) [4 points] Compute the density function $f_Z(z)$ of $Z := X + Y$. It is convenient to distinguish between $z < 0$, $0 \leq z \leq 2$, $2 < z \leq 4$ and $z > 4$.
(b) [3 points] Compute $P(Y > X^2)$.
(c) [3 points] Compute $\text{Cov}(X + Y, X)$

6. Let $X_1, X_2, \ldots$ be independent and exponentially distributed random variables with parameter 2.
(a) [3 points] Let Y be the number of the variables $X_1, X_2, \ldots, X_{625}$ that have an outcome greater than 1. Give an approximation of $P(Y > 80)$ that is based on the central limit theorem. Decide (and explain) whether you need to apply the continuity correction.
(b) [3 points] Let Z be the sixth index i such that $X_i > 1$ (so Z is the index i for which holds that both $X_i > 1$ and that there are exactly five indices $j < i$ with $X_j > 1$). Give the probability mass function and the expectation of Z.