Final Exam Probability Theory

May 27 2016, 15.15-17.15

- This exam consist of six exercises and a table. You can obtain 36 points. Your grade is given by (4+number of points)/4.
- You may use a simple calculator, but it is not allowed to use a graphical or a programmable calculator.
- Explain your answers clearly!

1. Let X and Y be continuous random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2x^2e^{-xy} & \text{if } 0 < x < 1 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Show that the marginal density function of X is given by

$$f_X(x) = \begin{cases} 2x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) [3 points] Show that $E(Y|X = x) = \frac{1}{x}$, for $0 < x < 1$.
(c) [2 points] Compute $E(Y)$.
(d) [3 points] Compute the covariance between X and Y.

2. Let X and Y be independent random variables and suppose that the density function of X is given by

$$f_X(x) = \begin{cases} e^{-x+1} & \text{if } x > 1, \\ 0 & \text{otherwise,} \end{cases}$$

and that the density function of Y is given by

$$f_Y(y) = \begin{cases} e^{y} & \text{if } y < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [4 points] Compute the density function $f_Z(z)$ of $Z := X + Y$. It is convenient to distinguish between $z \leq 1$ and $z > 1$.
(b) [3 points] Compute $Y > 2 - 2X$.

3. Let $X_1, X_2, \ldots, X_{225}$ be independent and identically distributed continuous random variables with $E(X_i) = 2$ and $\text{Var}(X_i) = 25$.
(a) [3 points] Give an approximation of $P(300 < X_1 + \cdots + X_{225} < 480)$ based on the central limit theorem.
(b) [3 points] Use the central limit theorem to find an a such that

$$P\left(\left|\frac{X_1 + X_2 + \cdots + X_{225}}{225} - 2\right| \leq a\right) \approx 0.85$$

holds.

4. [3 points] Six students attend a tutorial Linear Algebra. In the study guide, five homework problems are listed for this tutorial. Suppose that each student has prepared only one of these five problems and that this problem is chosen randomly and independently of the choice of the other students. Let X be the number of problems that are prepared by at least one student. Compute $E(X)$.

5. Let X and Y be independent random variables, both exponentially distributed with parameter 4.
(a) [3 points] Compute the density function of $W := \max(X, Y)$.
(b) [3 points] Compute the density function of $V := e^Y$.

6. [3 points] Let $X_1, X_2, \ldots, X_{100}$ be independent normally distributed random variables with $E(X_i) = 1$ and $\mathrm{Var}(X_i) = 4$. Let $V := \dfrac{X_1 + \cdots X_{64}}{64}$ and $W := \dfrac{X_{65} + \cdots X_{100}}{36}$. Compute $P(V < W)$.

Table

Area ϕ(x) under the standard normal curve to the left of x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998