Exam Probability Theory May 27, 2016

1a) For $0<x<1$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{\infty} 2x^2 e^{-xy}\,dy$$

$$= \left[-2xe^{-xy}\right]_0^{\infty} = 2x. \quad f_X(x) = 0 \text{ for } x \notin (0,1).$$

1b) $$E(Y|X=x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\,dy$$

$$= \int_{-\infty}^{\infty} y \cdot \frac{f_{X,Y}(x,y)}{f_X(x)}\,dy$$

$$= \int_0^{\infty} y \cdot \frac{2x^2 e^{-xy}}{2x}\,dy$$

$$= \int_0^{\infty} y \cdot x e^{-xy}\,dy \overset{*}{=}$$

$$= \left[y(-e^{-xy})\right]_0^{\infty} + \int_0^{\infty} e^{-xy}\,dy$$

$$= \left[-\frac{1}{x}e^{-xy}\right]_0^{\infty} = \frac{1}{x} \qquad (\text{for } 0<x<1)$$

$*$ or recognise the expectation of an exponentially distributed r.v. with $\lambda = x$ at this point.

1c) $$E(Y) = \int_{-\infty}^{\infty} E(Y|X=x) \cdot f_X(x)\,dx$$

$$= \int_0^1 \frac{1}{x} \cdot 2x\,dx = \int_0^1 2\,dx = 2.$$

1d) $$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$E(X) = \int_0^1 x \cdot 2x\,dx = \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3}.$$

$$E(XY) = \int_0^1 \int_0^\infty xy \cdot 2x^2 e^{-xy}\, dy\, dx$$

$$= \int_0^1 2x^2 \int_0^\infty xy \cdot e^{-xy}\, dy\, dx$$

$$= \int_0^1 2x^2 \cdot \frac{1}{x}\, dx$$

$$= \int_0^1 2x\, dx = \left[x^2\right]_0^1 = 1.$$

So $Cov(X,Y) = 1 - \frac{2}{3} \cdot 2 = -\frac{1}{3}$.

2) a) $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$

$$= \int_1^\infty e^{-x+1} f_Y(z-x)\, dx.$$

$z - x < 0$ gives $x > z$.

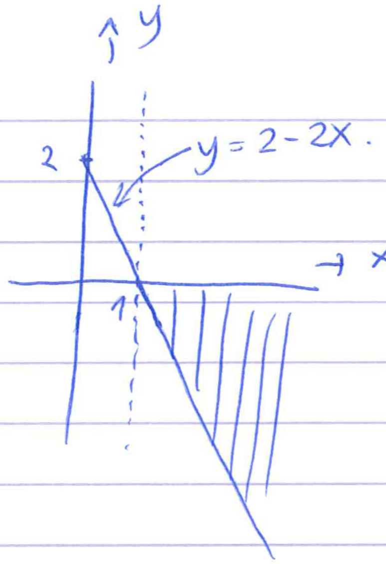
So for $z<1$: $f_Z(z) = \int_1^\infty e^{-x+1} \cdot e^{z-x}\, dx$

$$= \int_1^\infty e^{-2x+1+z}\, dx = \left[-\frac{1}{2} e^{-2x+1+z}\right]_1^\infty = \frac{1}{2} e^{z-1}.$$

For $z>1$: $f_Z(z) = \int_z^\infty e^{-x+1} \cdot e^{z-x}\, dx$

$$= \left[-\frac{1}{2} e^{-2x+1+z}\right]_z^\infty = \frac{1}{2} e^{-z+1}.$$

2b) $P(Y > 2-2X) =$



$$\int_1^\infty \int_{2-2x}^0 f_{X,Y}(x,y)\,dy\,dx$$

X,Y indep.

$$\stackrel{1}{=} \int_1^\infty \int_{2-2x}^0 e^{-x+1} \cdot e^y \,dy\,dx$$

$$= \int_1^\infty e^{-x+1} \left[e^y \right]_{2-2x}^0 dx$$

$$= \int_1^\infty \left(e^{-x+1} - e^{-x+1} \cdot e^{2-2x}\right) dx = \int_1^\infty \left(e^{-x+1} - e^{-3x+3}\right) dx$$

$$= \left[-e^{-x+1} + \frac{1}{3} e^{-3x+3} \right]_1^\infty = 1 - \frac{1}{3} = \frac{2}{3}.$$

3a) $P(300 < X_1 + \dots + X_{225} < 480) =$

$$= P\left(\frac{300 - 225 \cdot 2}{\sqrt{225}\sqrt{25}} < \frac{X_1 + \dots + X_{225} - 450}{15 \cdot 5} < \frac{480 - 450}{75}\right)$$

$$\stackrel{CLT}{\approx} \Phi\left(\frac{30}{75}\right) - \Phi\left(\frac{-150}{75}\right) = \Phi(0{,}4) - \Phi(-2)$$

$$= \Phi(0{,}4) - (1 - \Phi(2)) = 0{,}6554 - (1 - 0{,}9772)$$

$$= 0{,}6326.$$

3b) $P\left(-a \le \frac{X_1 + \cdots + X_{225} - 2\cdot 225}{225} \le a\right)$

$= P\left(\frac{-a\cdot 15}{5} \le \frac{X_1 + \cdots + X_{225} - 450}{15\cdot 5} \le \frac{15a}{5}\right)$

$\overset{CLT}{\approx} \Phi(3a) - \Phi(-3a) = \Phi(3a) - (1 - \Phi(3a))$

$= 2\,\Phi(3a) - 1 \approx 0{,}85$

So $\Phi(3a) \approx 0{,}925$, so $3a \approx 1{,}44$, so $a \approx 0{,}48$.

4) Write $X_i = \begin{cases} 1 & \text{if problem } i \text{ is chosen by at least one student} \\ 0 & \text{otherwise} \end{cases}$

Then $X = X_1 + \cdots + X_5$ and
$E(X) = E(X_1) + \cdots + E(X_5)$.

$E(X_i) = P(X_i = 1) = 1 - P(X_i = 0)$
$= 1 - P\left(\begin{array}{l}\text{none of the students} \\ \text{has chosen problem } i\end{array}\right)$
$= 1 - \left(\frac{4}{5}\right)^6$.

So $E(X) = 5\left(1 - \left(\frac{4}{5}\right)^6\right)$ $(\approx 3{,}69)$.

5a) Compute the cumulative distribution of W.

For $w<0$: $F_W(w)=0$

For $w\geq 0$: $P(W\leq w) = P(\max(X,Y)\leq w)$

$= P(X\leq w, Y\leq w)$

$\overset{\text{indep}}{=} P(X\leq w)\, P(Y\leq w)$

For $w\geq 0$: $P(X\leq w) = P(Y\leq w) = \int_0^w 4e^{-4t}\,dt$

$= \left[-e^{-4t}\right]_0^w = 1-e^{-4w}.$

So $F_W(w) = \begin{cases}(1-e^{-4w})^2 & \text{for } w\geq 0\\ 0 & \text{for } w<0\end{cases}$

and $f_W(w) = \begin{cases}2(1-e^{-4w})\cdot 4e^{-4w} & \text{for } w\geq 0\\ 0 & \text{otherwise.}\end{cases}$

5b) $V = e^Y$ so $Y=\ln(V)$

for $v\geq 1$: $f_V(v) = f_Y(\ln(v))\left|\frac{d\ln(v)}{dv}\right|$

$= 4e^{-4\ln v}\cdot\frac{1}{v}$

$= 4v^{-4}\cdot\frac{1}{v} = \frac{4}{v^5}$

for $v<1$: $f_V(v)=0$.

6) $E(V) = \frac{E(X_1)+\cdots+E(X_{64})}{64} = 1$

$Var(W) = \frac{1}{64^2}\left(Var(X_1)+\cdots+Var(X_{64})\right) = \frac{4}{64}$ (independence)

$V\sim N\left(1, \frac{4}{64}\right)$

$W \sim N\left(1, \frac{4}{36}\right)$

V and W are independent, so

$V - W \sim N\left(1-1, \frac{4}{36} + \frac{4}{64}\right)$

$\sim N\left(0, \frac{4}{36} + \frac{4}{64}\right)$

$P(V < W) = P(V - W < 0) = \frac{1}{2}.$