Faculteit de Exacte Wetenschappen	PDV (400163)
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No calculators. You can bring your book and notes. Explain what you do. You have to do 6 exercises. I will give equal weight to all exercises. The variable t is called time whenever it appears.

1. Let $u = u(x)$ be the unique bounded solution on the whole real line of the equation $-u'' + 2u' + 3u = \delta$. Derive a formula for $u(x)$.

2. The Fourier transform is by definition the appropriate extension of the map $f \to \hat{f}$ defined by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} f(x) \exp(-ikx) dx$$

for functions for which this limit exists. Explain why $f(x) = \exp(-\frac{1}{2}x^2)$ implies that $\hat{f}(k) = \exp(-\frac{1}{2}k^2)$.

3. The heat equation $u_t = u_{xx}$ has solutions of the form

$$u(t, x) = \frac{1}{t^\alpha} F(\frac{x}{\sqrt{t}})$$

with $F > 0$ and F integrable. Determine such a solution that has

$$\int_{-\infty}^{\infty} u(t, x) dx = 1$$

and $u(t, x) \to 0$ for $x \neq 0$ if $t \downarrow 0$.

4. Consider for $u = u(t, x)$ the first order partial differential equation (PDE)

$$u_t + c(x, u)u_x = h(x, u) \quad (x \in \mathbb{R},\ t \geq 0).$$

Derive the first order ordinary differential equations for $x = X(t)$ and $u = U(t)$ to impose so that for solutions $u(t, x)$ of the PDE it holds that $U(t) = u(t, X(t))$, and give an example of a such an equation that has a solution which is smooth at $t = 0$ and developes a shock in finite time.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic continuous function with partial Fourier sums $s_n(x)$. In the course it was shown that the sequence of averages

$$\sigma_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} s_n(x)$$

converges uniformly to $f(x)$ as $N \to \infty$. Explain directly from this result that

$$\int_{-\pi}^{\pi} |s_N(x) - f(x)|^2 dx \to 0$$

as $N \to \infty$.

6. Let $\epsilon \in \mathbb{R}$. Consider for $u = u(t, x)$ the equation

$$u_t = u_{xx}$$

with $0 < x < 1$. Given boundary conditions

$$u(t, 0) = 0 = \epsilon u_x(t, 1) + u(t, 1),$$

the PDE has solutions of the form $u(t, x) = T(t)X(x)$. For which ϵ do there exist nontrivial solutions $u(t, x) = T(t)X(x)$ with $T(t) \to \infty$ as $t \to \infty$?