Faculteit de Exacte Wetenschappen	PDV (400163)
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No calculators. You can bring your book and notes. Explain what you do. If you only do part 2 then you have to do the first 5 exercises. If you do the whole exam you have to do 6 exercises of your own choice. I will give equal weight to all exercises. The variable t is called time whenever it appears.

1. Let $u = u(x)$ be the unique bounded solution on the whole real line of the equation $-u'' + u = \delta$.
   (a) Derive a formula for $u(x)$. Hint: distinguish between $x < 0$ and $x > 0$.
   (b) Find the Fourier transform $\hat{u}(k)$ of $u(x)$ without using this formula.
   (c) Use your results to explain and conclude what the Fourier transform $\hat{f}(k)$ of $f(x) = \frac{1}{1+x^2}$ is.
   (d) In case you were not able to do (c) because you did not get (a) and/or (b): compute the Fourier transform $\hat{f}(k)$ of $f(x) = \frac{1}{1+x^2}$ using contour integration and residues. Explain which contour you take and how this choice depends on k.

2. The Fourier transform is by definition the appropriate extension of the map $f \to \hat{f}$ defined by
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} f(x) \exp(-ikx)dx$$
for functions for which this limit exists. Use appropriate rectangular contours in the complex plane containing the real interval $[-R, R]$ to explain why $f(x) = \exp(-\frac{1}{2}x^2)$ implies that $\hat{f}(k) = \exp(-\frac{1}{2}k^2)$.

3. Use the result of Exercise 2 to compute the Fourier transform $\hat{E}(t, k)$ of
$$E(t, x) = \frac{1}{2\sqrt{\pi t}} \exp(-\frac{x^2}{4t})$$
and evaluate the limits of
$$\int_{-\infty}^{\infty} E(t, x)\phi(x)dx \quad \text{and} \quad \int_{-\infty}^{\infty} \hat{E}(t, k)\phi(k)dk$$
as $t \downarrow 0$ for $\phi : \mathbb{R} \to \mathbb{R}$ continuous with compact support.

4. The heat equation $u_t = u_{xx}$ has solutions of the form

$$u(t,x) = \frac{1}{t^\alpha} F(\frac{x}{\sqrt{t}})$$

   (a) Assume that $F > 0$ and F is integrable. Explain why conservation of $\int_{-\infty}^{\infty} u(t,x)dx$ requires $\alpha = \frac{1}{2}$ and derive an ordinary differential equation (ODE) for $F(y)$ (note that $y = \frac{x}{\sqrt{t}}$, take $\alpha = \frac{1}{2}$).

   (b) What is the formula for $F(y)$ in the case that $u = E$ where E is as in Exercise 3? Verify that this F solves the ODE you derived.

5. Consider $u_t + u_{xxx} = 0$ with initial data $u_0(x) = u(0,x)$. Explain why the Fourier transform $\hat{u}(t,k)$ of $u(t,x)$ should be given by

$$\hat{u}(t,k) = \widehat{u}_0(k) e^{ik^3 t}$$

The inversion formula applied to the case that $\widehat{u}_0(k) = \hat{\delta}(k) = \frac{1}{\sqrt{2\pi}}$ defines a solution formula

$$u(t,x) = \frac{1}{(3t)^{\frac{1}{3}}} \mathrm{Ai}(\frac{x}{(3t)^{\frac{1}{3}}})$$

in which

$$\mathrm{Ai}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\xi x + \frac{x^3}{3})} dx.$$

Use the methods in the first part of the file Airy.pdf to determine the asymptotic behaviour of $\mathrm{Ai}(\xi)$ for $\xi \to -\infty$. That is: find a function $f(\xi)$ such that

$$\lim_{\xi \to -\infty} \frac{\mathrm{Ai}(\xi)}{f(\xi)} = 1.$$

6. Consider for $u = u(t,x)$ the first order partial differential equation (PDE)

$$u_t + c(x,u)u_x = h(x,u) \quad (x \in \mathbb{R},\ t \geq 0).$$

   (a) Derive the first order ordinary differential equations for $x = X(t)$ and $u = U(t)$ to impose so that for solutions $u(t,x)$ of the PDE it holds that $U(t) = u(t, X(t))$.

   (b) Take $c(x,u) = u$ and $h(x,u) = 0$, consider the solution of the PDE that satisfies the initial condition $u(0,x) = \exp(-x^2)$. Explain why the solution develops a shock in finite time. At what time does the shock appear?

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic continuous function with partial Fourier sums $s_n(x)$. In the course it was shown that the sequence of averages

$$\sigma_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} s_n(x)$$

converges uniformly to $f(x)$ as $N \to \infty$. Explain directly from this result that

$$\int_{-\pi}^{\pi} |s_N(x) - f(x)|^2 dx \to 0$$

as $N \to \infty$.

8. Let $\beta \in \mathbb{R}$. Consider for $u = u(t, x)$ the equation

$$u_t = u_{xx}$$

with $0 < x < 1$. Given boundary conditions

$$u(t, 0) = 0 = u_x(t, 1) + \beta u(t, 1),$$

the PDE has solutions of the form $u(t, x) = T(t)X(x)$. For which β do there exist nontrivial solutions $u(t, x) = T(t)X(x)$ with $T(t) \to \infty$ as $t \to \infty$?