Faculteit der Exacte Wetenschappen	Partiële Differentiaalvergelijkingen (400163)
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Vrije Universiteit	2 uur

No formula sheets, no calculators. Explain what you do. Do **exercise 1 or 2** but not both (3 points). Do **exercise 3 and 5** (2 points each) **or** do **exercise 4** (4 points). Do **exercise 6 or 7** but not both (2 points). Do **exercise 8** (1 point). Grade: total score plus 1, but not more than 10.

1. Consider for $u = u(t, x)$ the first order partial differential equation (PDE)

$$u_t + c(x, u)u_x = h(x, u),$$

with initial condition at $t = 0$ given by

$$u(0, x) = f(x) > 0.$$

Here $x \in \mathbb{R}$ and $t \geq 0$. We assume that $(x, u) \to c(x, u)$ and $(x, u) \to h(x, u)$ define smooth functions on $\mathbb{R}^2$, and that $f : \mathbb{R} \to \mathbb{R}$ is also smooth.

(a) Let $x = X(t)$ be a smooth curve in the t, x-plane and assume that $u = u(t, x)$ is a smooth solution of the (PDE). Let $U(t) = u(t, X(t))$. Derive an equation of the form $\dot{X} = ..$ in terms of X and U that leads to an equation of the form $\dot{U} = ..$ in terms of X and U.

(b) Take $c(x, u) = c(x) = x(1 - x)$. The differential equation for $X(t)$ then decouples from the equation for $U(t)$. Its solutions define all the characteristic curves in the t, x-plane. Determine the general solution of the differential equation for $x = X(t)$ in the range $0 < x < 1$.

(c) If the characteristic curve through a given point (t, x) intersects the vertical axis in the t, x-plane we denote the point of intersection by $(0, k)$. Derive an equation for k in terms of the coordinates t and x of the given point if $0 < x < 1$.

(d) Take again $c(x, u) = c(x) = x(1 - x)$ and $h(x, u) \equiv 0$. Give a second equation for the value of $u = u(t, x)$ in the same given point (t, x) which involves k and $f(k)$. Determine $u = u(t, x)$ when $t > 0$ and $0 < x < 1$.

(e) Now take $c(x, u) = c(x) = x(1 - x)$ and $h = h(u) = -u^2$. Then the second equation in Part 1d above has to be replaced by another equation for the value of $u = u(t, x)$ in the same given point (t, x). Determine again $u = u(t, x)$ when $t > 0$ and $0 < x < 1$.

(f) For which (t, x) with $t > 0$ are the solution formula's you found also valid?

2. Consider for $u = u(t, x)$ the first order partial differential equation (PDE)

$$u_t + uu_x = 0,$$

with initial condition at $t = 0$ given by

$$u(0, x) = f(x) = \frac{1}{1 + e^x}$$

Here $x \in \mathbb{R}$ and $t \geq 0$. The graph $u = f(x)$ has $(x, u) = (0, \frac{1}{2})$ as a point of symmetry and may be described by $x = g(u)$. The function G defined by

$$G(u) = \int_{\frac{1}{2}}^{u} g(s)ds$$

is symmetric around $u = \frac{1}{2}$.

(a) Explain why the smooth part of the solution is contained in the graph $x = g(u) + ut$. Observe that $(x, u) = (\frac{t}{2}, \frac{1}{2})$ remains a point of symmetry!

(b) For which t is the point of symmetry $(x, u) = (\frac{t}{2}, \frac{1}{2})$ no longer on the graph of the smooth part of the solution?

(c) Explain why the equal area rule implies that for every such t the shock is described by

$$u_- = \frac{1}{2} + \epsilon \qquad u_+ = \frac{1}{2} - \epsilon$$

with ϵ depending on t. Express t in ϵ. What is the x-location of the the shock (in Olver's notation, what is $\sigma(t)$)?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic continuous function with partial Fourier sums $s_n(x)$. In the course it was shown that the sequences of averages

$$\sigma_N(x) = \sum_{n=0}^{N} s_n(x)$$

converges uniformly to $f(x)$ as $N \to \infty$, i.e.

$$\max_{x \in \mathbb{R}} |\sigma_N(x) - f(x)| \to 0.$$

Use the inner product structure to derive from this result that

$$\int_{-\pi}^{\pi} |s_N(x) - f(x)|^2 dx \to 0$$

as $N \to \infty$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd and 2π-periodic piecewise smooth function. The Fourier series of f is given by

$$\sum_{n=1}^{\infty} b_n \sin nx.$$

(a) To make sure you use the right formula's for the Fourier coefficients: derive the integral formula's for b_n in the case that

$$f(x) = \sum_{n=1}^{N} b_n \sin nx$$

for some integer $N > 0$.

(b) Use the case that $f(x) = \pi - x$ for $0 < x < \pi$ to compute

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

in terms of π by using the explicit values of b_n. Explain why your answer is correct. Hint: the answer involves $\int_0^{\pi} f(x)^2 dx$.

(c) For the same f, let

$$s_N(x) = \sum_{n=1}^{N} b_n \sin nx.$$

Show that $s'_N(x) + 1$ is a multiple of

$$D_N(x) = \frac{\sin(N + \frac{1}{2})x}{\sin \frac{x}{2}}$$

by using the complex formula's for $\cos nx$ and the values of b_n.

(d) Examine the integral

$$g_N(x) = \int_x^{\pi} D_N(s) ds$$

by first sketching the graph of D_N for N not too small. Hint: the denominator is monotone in x. Use this graph to give a sketch of the graph of g_N for $0 < x < \pi$ and explain why $g_N(x) \to 0$ as $N \to \infty$ for $0 < x < \pi$.

5. Use the Fourier series of $f(x) = x^2$ with $|x| \leq \pi$ to determine

$$\sum_{n=0}^{\infty} \frac{1}{n^4}.$$

6. Consider for $u(t,x)$ the wave equation

$$u_{tt} = u_{xx}$$

The initial condition at $t = 0$ is

$$u(0,x) = f(x), \quad u_t(0,x) = 0,$$

with $f : \mathbb{R} \to \mathbb{R}$ 2π-periodic and smooth. Therefore f equals its complex Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

What is the corresponding Fourier series formula for the solution $u(t,x)$? Use complex notation for the time-dependent part to rederive the d'Alembert formula.

7. Consider for $u(t,x)$ the inhomogoneous wave equation

$$u_{tt} = u_{xx} + f(t,x)$$

in which f is a smooth function. The initial condition at $t = 0$ is

$$u(0,x) = 0, \quad u_t(0,x) = 0,$$

Change to variables $\xi = x - t$ and $\eta = x + t$ and rewrite the equation for $v(\xi,\eta) = u(t,x)$ with $g(\xi,\eta) = f(t,x)$. Explain why $v = v_\xi = v_\eta = 0$ on the diagonal $\eta = \xi$.

8. Let $\beta \in \mathbb{R}$. Consider for $u = u(t,x)$ the equation

$$u_t = u_{xx}$$

with $0 < x < 1$. Given boundary conditions

$$u_x(t,0) = 0 = u_x(t,1) + \beta u(t,1),$$

the PDE has solutions of the form $u(t,x) = T(t)X(x)$. The only way to have u satisfy $u_x = 0$ in $x = 0$ is to have $X(x) = \cos(\mu x)$ or $X(x) = \cosh(\mu x)$ (up to a mulitplicative constant) with $\mu \geq 0$.

   (a) Which of these two types solutions have $T(t) \to \infty$?

   (b) For which β do such solutions with $X(x) = \cosh(\mu x)$ and $\mu > 0$ exist which also satisfy the other boundary condition $u_x + \beta u = 0$ in $x = 1$?