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| Faculteit der Exacte Wetenschappen | Partial Differential equations |
| Afdeling Wiskunde | 15:15 – 17:15 |
| Vrije Universiteit | 24-03-2010 |

Books, notes and calculators are not allowed. Please work tidy & justify your answers. Good luck!

You might use the following relations:

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 \,, \ \forall n, m \in \mathbb{N} \,, \ n \neq m \,.$$

1. Let $f(x)$ be a $2p$–periodic, piecewise smooth function given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x) \,.$$

   (a) Set $p = \pi$ and show that

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \,, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \,, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \,.$$

   (b) Consider the *mean square error* defined as $E_N = \frac{1}{2p} \int_{-p}^{p} (f(x) - s_N(x))^2 dx$, where $s_N = a_0 + \sum_{n=1}^{N} (a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x)$ are the $N^{th}$-partial sums. Assume $f(x)$ square integrable on $[-p \,, p]$.
   Prove the Bessel's inequality

$$a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{2p} \int_{-p}^{p} f(x)^2 dx \,.$$

   Prove the Parseval's identity

$$\frac{1}{2p} \int_{-p}^{p} f(x)^2 dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \,.$$

2. Given $f(x) = \pi - |x|$ on $[-\pi \,, \pi]$

   (a) compute its Fourier coefficients and write the Fourier series.

P.T.O.

(b) Compute the sum $\sum_{n=0}^{\infty} \frac{1}{(1+2n)^2}$;

(c) Compute the sum $\sum_{n=0}^{\infty} \frac{1}{(1+2n)^4}$ [Hint: use Parseval's identity].

3. Let $u = u(x,t)$ and let $k > 0$ be a real parameter; consider the following PDEs

$$u_{tt} + 2ku_t = u_{xx}\,, \quad 0 < x < \pi\,,$$

with boundary values $u(0,t) = u(\pi,t) = 0$.

(a) Compute all solutions of the form $u(x,t) = T(t)X(x)$.

(b) For which values of $k$ are there solutions which oscillate in time?

4. Solve the Dirichlet problem

$$\nabla^2 u = 0\,, \quad 0 < r < 1\,, \quad 0 < \theta < 2\pi\,,$$

with $u(1,\theta) = 1 + \sin 2\theta$.

**Score:**

Exercise 1. and 2. are worth 25 points each, Exercises 3. is worth 20 points, Exercise 4. is worth 10 points.

**Final score:**

$$\frac{1}{10} \cdot \#\ \text{points}$$