Optimization and Multiagent Systems
Exam

Provide formal arguments for your claims. Be concise. If the space in the boxes is not sufficient, use additional sheets and clearly indicate which questions your answers belong to.

Question 1 (4 + 12 points)

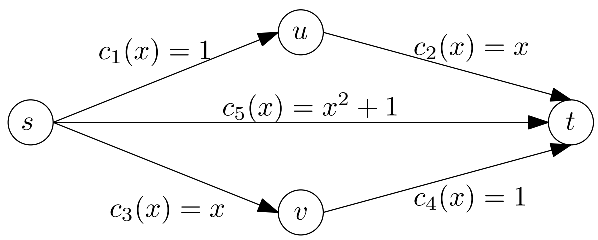

Figure 1: Network for Question 1

For a non-atomic selfish routing game, consider the road network depicted in Figure 1 with a single commodity (type of player) with traffic rate 1, origin s and destination t. Before building a new direct road e_5 from s to t with cost function $c_5(x) = x^2 + 1$, agents could only travel either via node u (taking edges $e_1 = (s, u)$ and $e_2 = (u, t)$ with costs $c_1(x) = 1$ and $c_2(x) = x$) or via node v (taking edges $e_3 = (s, v)$ and $e_4 = (v, t)$ with costs $c_3(x) = x$ and $c_4(x) = 1$). For the following questions, the social cost is the standard utilitarian cost $\sum_{e \in E} c_e(x) \cdot x$.

a) Did adding the new road e_5 increase or decrease the social cost in the network in equilibrium? Give an argument for your answer.

b) Calculate the price of anarchy in the network after the introduction of the new road e_5.
Hint: In order to make calculations easier, you can re-design this network as an equivalent Pigou-like network with two edges and two nodes s, t and argue about the flow in that network.

Solution:

a) The total social cost went down. Before the addition, agents split equally between the two paths $s - u - t$ and $s - v - t$ for a travel time of $\frac{3}{2}$ each. After construction, agents move equally to the new edge until the costs on all paths is $\frac{5}{4}$ (which happens when half of the flow goes over the middle edge and $\frac{1}{4}$ over the other edges). Since the cost on the paths $s - u - t$ and $s - v - t$ goes down and all paths have the same cost in equilibrium, the total social cost went down.

b) Figure 2 shows the Pigou-like network with two edges.

- We already determined the Nash flow in the previous exercise. We can demonstrate it again for the two-edge network: The Nash equilibrium flow is given for $x^2 + 1 = \frac{1}{2}(1 - x) + 1$ which holds for $x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{4}$ and hence $x = \frac{1}{2}$. Each path has a cost of 1.25 and hence the social cost of the flow is 1.25.
- For the social optimum, we can derive the Nash equilibrium flow in the network with the marginal cost functions. This leads to $3x^2 + 1 = (1-x)+1$, which holds for $x_{1,2} = \frac{-1 \pm \sqrt{1+12}}{6}$ and hence $x = \frac{\sqrt{13}-1}{6}$. The social cost in the original network is then

$$\left((\frac{\sqrt{13}-1}{6})^2 + 1 \right) \cdot \frac{\sqrt{13}-1}{6} + \left(\frac{7-\sqrt{13}}{12} + 1 \right) \cdot \frac{7-\sqrt{13}}{6} \approx 1.242$$

- The price of anarchy therefore is 1.0065.

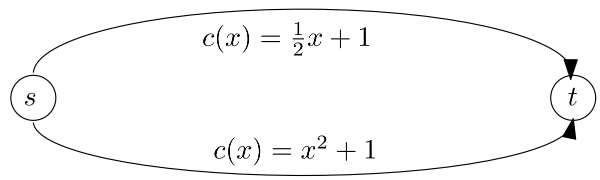

Figure 2: Network for Question 1

Question 2 (10 points)

Give an ordinal potential function for the following 2 player bimatrix game where both the row and the column player have three actions A, B and C.

	A	B	C
A	80, 10	5, 5	10, 8
B	41, 12	13, 24	8, 6
C	20, 5	2, 20	9, 15

Solution: An example of an ordinal potential

	A	B	C
A	10	70	55
B	90	0	110
C	110	75	90

Question 3 (10 points)

Prove that the price of anarchy for the scheduling game $P2(LPT)|u_j = -C_j|C_{\max}$ (i.e., the sequencing model on $m = 2$ machines with local policy that jobs are sequenced non-increasing in their processing times on each machine) is at least $\frac{7}{6}$.

Solution: We give an example to show the lower bound. Consider 5 jobs as follows: Jobs $1, 2$ have processing time 3 and jobs $3, 4, 5$ have processing time 2.

- Optimum solution: Jobs 1,2 are scheduled on machine 1 and jobs 3,4,5 are scheduled on machine 2. The makespan is 6.
- Nash equilibrium: Jobs 1,3,4 are scheduled on machine 1 and jobs 2,5 are scheduled on machine 2. The makespan is 7.

Question 4 (8+4+10 points)

Consider the following Bayesian game with players Alice and Bob. Both players can play the actions C and D. Alice is of a fixed type θ_A, but Bob can be either *balanced* with probability p or *unbalanced* with probability $1 - p$. In the following bimatrix representation of the utility, Alice is the row player and Bob is the column player. In the case that Bob is balanced, the utilities are as follows:

	C	D
C	2,2	0,0
D	0,0	1,1

In the case that Bob is unbalanced, the utilities are as follows:

	C	D
C	2,2	0,3
D	0,0	1,1

For all the following questions, state the strategies and provide a proof that they are indeed equilibrium strategies.

a) Suppose that $p < \frac{1}{3}$. What is the only ex-ante Bayes-Nash equilibrium in this game? **Hint:** Start by reasoning what Bob will do when he is unbalanced.

b) Suppose that $p = \frac{1}{3}$. Give a pure-strategy ex-ante Bayes-Nash equilibrium that is different from the one in the previous answer.

c) Let $p > \frac{1}{3}$. Find a mixed-strategy ex-ante Bayes-Nash equilibrium.

Solution:

a) If Bob is unbalanced, he will always play D since this is his dominant strategy. Let p be the probability that Bob is balanced and $1 - p$ be the probability that Bob is unbalanced. Let a be the probability that Bob plays C when he is balanced and $1 - a$ be the probability that Bob plays D when he is balanced. For Alice, her expected utility for playing C is $2ap$, and her expected utility for playing D is $(1 - a)p + (1 - p) = 1 - ap$. If $p < \frac{1}{3}$ then the expected utility for playing C is less than $\frac{2}{3}a \leq \frac{2}{3}$ while for D it is more than $1 - \frac{1}{3}a \geq \frac{2}{3}$. Therefore, Alice maximizes her utility by playing D, regardless of a. If Alice always plays D, then Bob also should play D when he is balanced since that is his best response to Alice playing D. Therefore, the unique Bayes-Nash equilibrium is reached by Alice playing D and Bob playing D no matter his type.

b) With the previous analysis, if Bob plays C with probability $a = 1$ when balanced, then Alice is indifferent between playing C and D since her expected utility for playing C is $2ap = \frac{2}{3}$ and for playing D is $1 - ap = \frac{2}{3}$. Therefore, a pure-strategy Bayes Nash equilibrium is reached for Alice always playing C and Bob playing C when he is balanced and D when he is unbalanced.

c) In order for Alice to play a mixed strategy, Bob needs to play C with a probability a such that Alice is indifferent between playing C and D. Therefore, $2ap = 1 - ap$ has to hold, or $a = \frac{1}{3p}$. Similarly, Alice needs to play C with probability q such that Bob is indifferent between playing C and D in the case that he is balanced (he will still always play D when he is unbalanced). Therefore, it needs to hold that $2q = 1 - q$ or $q = \frac{1}{3}$. Therefore, if Alice plays C with probability $\frac{1}{3}$, Bob is indifferent between playing C and D when he is unbalanced. The mixed-strategy Bayes Nash equilibrium for $p > \frac{1}{3}$ can therefore be described as follows: Alice plays C with probability $\frac{1}{3}$ and D with probability $\frac{2}{3}$. If Bob is balanced, he plays C with probability $\frac{1}{3p}$ and D with probability $\frac{3p-1}{3p}$. If Bob is unbalanced, he plays D.

Question 5 (12 + 4 points)

a) Consider a first-price auction with two risk-affine bidders $i = 1, 2$. A winning bidder i has utility function $u_i = (v_i - b_i)^2$ where b_i is the price that bidder i needs to pay (also its bid). Values v_1, v_2 are drawn independently from the uniform distribution over $[0, 1]$. Show that bidding $(\frac{1}{3}v_1, \frac{1}{3}v_2)$ constitutes a Bayes-Nash equilibrium in this auction.

b) Assume that only bidder 1 has this risk-affine utility function while bidder 2 has a risk-neutral utility function $u_2 = (v_2 - b_2)$. Explain whether and how the equilibrium strategies of bidders 1 and 2 would change in the first-price auction as compared to the case where both bidders have identical (either both risk-affine or both risk-neutral) utility function.

Solution:

a) We will show that $(\frac{1}{3}v_1, \frac{1}{3}v_2)$ constitutes a Bayes Nash equilibrium strategy profile. We prove this by fixing the strategy of bidder 2 to $\frac{1}{3}v_2$ and show the best response strategy s_1 of bidder 1. Since the value of v_2 is drawn from a uniform distribution, the expected utility u_1 of bidder 1 is

$$\begin{aligned} E[u_1] &= \int_0^1 u_1 dv_2 \\ &= \int_0^{3s_1} (v_1 - s_1)^2 dv_2 \\ &= 3 \cdot (v_1 - s_1)^2 \cdot s_1 \end{aligned}$$

Take the derivative for s_1 and set to 0.

$$-6(v_1 - s_1)s_1 + 3(v_1 - s_1)^2 = 0$$
$$3s_1^2 - 4v_1 s_1 + v_1^2 = 0$$
$$s_1 = \frac{4 \pm 2}{6} v_1$$

The second derivative is $6s_1 - 4v_1$ and hence negative for $s_1 = \frac{1}{3}v_1$, but positive for $s_1 = v_1$. Therefore $s_1 = \frac{1}{3}v_1$ is a maximum for the function and the agent maximizes its utility by bidding $s_1 = \frac{1}{3}v_1$. Due to symmetry this also holds for the second bidder and $(\frac{1}{3}v_1, \frac{1}{3}v_2)$ is a Bayes-Nash equilibrium.

b) The strategies would not change since playing $s_1 = \frac{1}{3}v_1$ is the best response against any bidding strategy in which bidder 2 plays a fraction of its value. Similarly, this also holds for a risk-neutral bidder 2, for whom bidding $s_2 = \frac{1}{2}v_2$ is a best response for bidder 1 playing $\frac{1}{3}v_1$.

Question 6 (6 + 8 points)

Consider the following extension of the sponsored search setting of bidders with quasi-linear utility functions: Each bidder $i \in I$ now has a publicly known quality β_i in addition to the private valuation v_i per click. As usual, slot j has a click-through rate of α_j with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$. Due to the quality of the bidder, the *adjusted click-through rate* of a slot is $\beta_i \alpha_j$ and hence bidder i derives a value of $\alpha_j \cdot \beta_i \cdot v_i$ from obtaining the j-th slot.

a) Describe a direct strategy-proof mechanism that maximizes the utilitarian social welfare for this setting where bidders $i \in I$ report their valuations $\hat{v}_i$. Describe how you assign bidders to slots and define the payments for each bidder. Define the payment for a winning bidder i as a function of $\alpha_1, \ldots, \alpha_k$ and $\beta_h, \hat{v}_h$ for $h \in I$. You do not need to provide a proof that the mechanism is strategy-proof.

b) Consider the example with $k = 3$ slots with click-through rates $\alpha_1 = 10, \alpha_2 = 5, \alpha_3 = 2$ and four bidders with valuations and qualities given in Table 1:

i	v_i	β_i
1	8	0.375
2	6	1
3	5	0.8
4	2	0.5

Table 1: Valuations for Question 6b

Calculate the welfare-maximizing allocation and VCG-payments for the bidders in this example.

Solution:

a) Allocation: Sort the bidders non-decreasing in $\beta_i \cdot \hat{v}_i$. Then assign the slot with the l-th highest click-through rate to the agent with the l-th highest $\beta_i \cdot \hat{v}_i$. Let w.l.o.g. the bidders be sorted like this. Then bidder i for $i \leq k$ has to make the following payment:

$$\begin{aligned} p_i &= \sum_{j=1}^{i-1} \beta_j \cdot \hat{v}_j \cdot \alpha_j + \sum_{j=i+1}^{k+1} \beta_j \cdot \hat{v}_j \cdot \alpha_{j-1} - \left(\sum_{j=1}^{i-1} \beta_j \cdot \hat{v}_j \cdot \alpha_j + \sum_{j=i+1}^{k} \beta_j \cdot \hat{v}_j \cdot \alpha_j \right) \\ &= \sum_{j=i+1}^{k} \beta_j \cdot \hat{v}_j \cdot (\alpha_{j-1} - \alpha_j) + \beta_{k+1} \cdot \hat{v}_{k+1} \cdot \alpha_k \end{aligned}$$

where $\hat{v}_j$ is defined as 0 for all $j > |I|$.

b) The welfare-maximizing allocation is to assign slot 1 to bidder 2, slot 2 to bidder 3, and slot 3 to bidder 1 for a welfare of $60 + 20 + 6 = 86$. The prices are as follows:

- $p_1 = 57 - 26 = 31$
- $p_2 = 77 - 66 = 11$
- $p_3 = 82 - 80 = 2$

Question 7 (12 points)

Consider a two-sided matching markets with agents m_1, m_2, m_3 on the one side and agents w_1, w_2, w_3 on the other side. Agents have the following preferences of being matched to an agent of the other side:

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$
w_2	w_3	w_2
w_3	w_2	w_1
w_1	w_1	w_3

$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
m_3	m_2	m_2
m_2	m_3	m_1
m_1	m_1	m_3

Prove that there exists only one stable matching given these preferences and state that stable matching.

Solution:

The men-proposing DA leads to

Step	w_1	w_2	w_3
1		m_1, $\boxed{m_3}$	$\boxed{m_2}$
2		$\boxed{m_3}$	m_1, $\boxed{m_2}$
3	$\boxed{m_1}$	$\boxed{m_3}$	$\boxed{m_2}$

and the matching $(m_1, w_1), (m_2, w_3), (m_3, w_2)$.

The women-proposing DA leads to

Step	m_1	m_2	m_3
1		$w_2, \boxed{w_3}$	$\boxed{w_1}$
2		$\boxed{w_3}$	$\boxed{w_2}, w_1$
3		$w_1, \boxed{w_3}$	$\boxed{w_2}$
4	$\boxed{w_1}$	$\boxed{w_3}$	$\boxed{w_2}$

and the matching $(m_1, w_1), (m_2, w_3), (m_3, m_2)$.

Men get their best possible woman in the man-proposing DA and the worst possible woman in the woman-proposing DA. Since the assignments are the same, there cannot be any other woman possible for any man and hence there is no other stable matching.