Optimization and Multiagent Systems Exam

E_EORM_OMS, School of Business and Economics, Vrije Universiteit Amsterdam
Prof. Stefan Waldherr, Ruben Brokkelkamp

Provide formal argument for your claims. Be concise.
27 May 2021

Question 1

Consider a normal-form game with two players. Player 1 (the row player) can perform actions A and B, while player 2 (the column player) can perform actions C and D. For $a \geq 0$ the following bimatrix gives the utilities of the players for the given action profiles:

	C	D
A	1-a,1+a	1,0
B	0,0	0,1+a

Determine the set of all $a \geq 0$ for which there exist pure Nash equilibria and specify the equilibria. For the $a \geq 0$ for which there does not exists a pure Nash equilibrium, determine the set of mixed Nash equilibria.

Solution: For any a, the column player wants to deviate from (B, C) and (A, D) and the row player wants to deviate from (B, D), therefore these are never pure Nash equilibria. For $a > 1$, the row player also wants to deviate from (A, C) and therefore there exists no pure Nash equilibrium for $a > 1$. For $a \leq 1$, the unique Nash equilibrium is (A, C).

The mixed strategy Nash equilibrium for $a > 1$ can be found as follows. We look for the probability p with which the column player plays C such that the column player is indifferent between playing A and B. This is the case for $(1 - a)p + (1 - p) = 0$ or $p = \frac{1}{a}$. Then we look for the probability q with which the row player plays A such that the column player is indifferent between playing C and D. This is the case for $(1 + a)q = (1 + a)(1 - q)$ or $q = \frac{1}{2}$. Hence, the mixed strategy Nash equilibrium is $(0.5, 0.5), (\frac{1}{a}, \frac{a-1}{a})$.

Grading Scheme:

- 12 points
  - 4 points for identifying the pure NE (minus points if errors in the bounds, depending on severity)
  - 4 points for providing the mixed strategy of each player (minus points for incomplete/erroneous calculation)

Question 2

Consider a normal-form game with two players. Player 1 (the row player) and player 2 (the column player) can both perform actions A and B. For $a, b, c, d \in \mathbb{R}$ the following bimatrix gives the utilities of the players for the given action profiles:

	A	B
A	a,a	c,d
B	d,c	b,b

Show that for any $a, b, c, d \in \mathbb{R}$, this game must have a pure strategy Nash equilibrium.

Solution: Assume there is no pure strategy Nash equilibrium. In this case, the best response algorithm must either cycle $(A, A) \to (A, B) \to (B, B) \to (B, A) \to (A, A)$ or $(A, A) \to (B, A) \to (B, B) \to (A, B) \to (A, A)$.

For both cycles it must then hold that $a < d$ and $a > d$, which is of course not possible.

Other option. If $a \geq d$, then we see that (A, A) is a PNE. If $b \geq c$, then (B, B) is a PNE. Otherwise if both $a < d$ and $b < c$, then (B, A) is a PNE.

Grading Scheme:

- 8 points
  - 8 points for one of the solutions above
  - Alternative: -1 point for every wrong/forgotten case when doing case distinction on a, b, c, d

Question 3

Consider a first-price auction for a single item with n bidders. The bidders' valuation of the object are public information, all different, and all positive in a way such that $v_1 > v_2 > \ldots > v_n$.

Proof or disprove: For each bidder k there exists Nash equilibrium of bids $b_1, \ldots, b_n$ such that bidder k wins the auction.

Solution: This is not true. Consider bidder $k > 1$ winning the auction with a bid b_k. Then the bidder needs to submit a bid that is higher than the bid of bidder 1. Then, either the bid is $b_1 \leq b_k < v_1$ in which case bidder 1 would deviate from his strategy and bid slightly more than b_k to increase his utility from 0 to a positive value. Or the bid is $b_k > v_1 \geq b_1$ in which case bidder k makes a loss and would want to deviate from bidding b_k to bidding 0 instead, as this would improve his utility to 0. In any case, this would not be a Nash equilibrium.

Grading Scheme:

- 9 points
  - 2 points for identifying there is no NE.
  - 7 for a proof that there is no NE (minus points for missing arguments, partial points for partial solutions, see individual comments)

Question 4

Consider a second-price auction for a single item with n bidders. The bidders' valuation of the object are public information, all different, and all positive in a way such that $v_1 > v_2 > \ldots > v_n$.

Proof or disprove: For each bidder k there exists Nash equilibrium of bids $b_1, \ldots, b_n$ such that bidder k wins the auction.

Solution: This is true. k bidding v_1 and every other bidder bidding 0 is a Nash equilibrium in which bidder k gets the item. No other bidder would want to deviate from bidding 0, since they cannot achieve a higher utility than 0. Also, k does not want to deviate since she cannot improve upon her utility of v_k.

Grading Scheme:

- Total: 9 points
  - 2 points for identifying there is an NE or arguing that truthful reporting leads to a NE in which b_1 wins the auction.
  - 7 for a providing the NE in which other bidders win the auction (minus points for missing arguments, partial points for partial solutions, see individual comments)

Question 5

A thousand tourists are visiting a city. To explore the city, they can either choose to book a ticket with touring company E or with company W. Both have a nice private bus, but the private bus of company E is only allowed to drive through the eastern part of the city while the private bus of company W can only drive on the west side. For the other side of the city, the tour guide guides tourists using public transport.

If e tourists choose to use the private bus of company E, it takes those tourists $e/20$ minutes to explore the eastern part of the city with the bus. If w tourists choose to use the private bus of company W, it takes those tourists $w/20$ minutes to explore the western part of the city with the bus.

Exploring the western part with public transport takes a fixed 55 minutes while exploring the eastern part with public transport takes a fixed 65 minutes, regardless of the number of tourists that use this mode of transportation..

Each tourist can only choose one of the two companies to explore the city and hence can only ride on one of the two private busses, while the tourist must use public transportation to explore the other part of the city. Each tourist wants to explore both parts of the city and will choose the option that minimizes their time for doing so.

(a) How many of the 1000 tourists will choose company E and how many will choose W in an equilibrium? How long does each tour take? Justify your answer by demonstrating that it is indeed in equilibrium.

Good news! Touring companies E and W are merging and tourists can now mix and match between the private busses and public transport. Tourists can decide to do both parts with the private busses, both parts with the public transport, visit the east with a private bus and the west with public transport, or vice versa.

(b) How will the 1000 tourists divide themselves over the four possible combinations in equilibrium? Justify your answer by demonstrating that it is indeed in equilibrium.

Solution:

(a) In an equilibrium no tourist wants to switch to the other tour, hence both tours should take the same time. Suppose x tourists choose tour company E, then $1000 - x$ choose company W. We have to solve

$$\frac{x}{20} + 55 = \frac{1000 - x}{20} + 65 \iff \frac{x}{10} = 60$$

which is solve for $x = 600$. Thus 600 tourists will choose touring company E and 400 will choose W.

Each tour will take $600/20 + 55 = 400/20 + 65 = 85$ minutes.

(b) Even if all 1000 tourists do the east part of the city with the private bus for a total of 1000/20 = 50 minutes, it is still faster than doing it with public transport which takes 55 minutes. Same holds for the west part of the city since $1000/20 > 65$. Thus all tourists will do both parts of the city with a private bus.

All tourists will take $1000/20 + 1000/20 = 100$ minutes.

Grading Scheme:

(a) 8 points

- 2 points for realizing that in equilibrium the two options should have the same time OR mentioning that for the equilibrium nobody has a reason to deviate since it will increase travel time
- 5 points for correct 600/400 calculation
- 1 point for mentioning that each tour takes 85 minutes

(b) 8 points

- 6 points for arguing that all tourists will take both private busses
- 2 points for mentioning all tourists take 100 minutes.
- 2 points if (wrongly) argued that everybody taking both private busses results in a total travel time of $500/20 + 500/20 = 50$ and is therefore shortest.

Question 6

Consider a situation with countries C. Each country has the following two choices: Either it agrees to set industry standards in order to control its CO2 emissions, or not. Mixed strategies are not allowed. Agreeing to set industry standards invokes a cost of 3 for the country that sets the standards. Each country that does not agree to set standards to control its emissions adds 1 to the cost of all countries, including itself.

More formally, every country $i \in C$ has a strategy set $S_i = \{\text{Yes}, \text{No}\}$. Given a strategy profile $s \in S_1 \times \cdots \times S_n$ we define the set of all countries that choose the strategy "no" as:

$$N(x) = \{j \in C \mid s_j = \text{No}\}$$

The cost of country i is given by

$$c_i(x) = \begin{cases} 3 + |N(x)| & \text{if } s_i = \text{Yes}\} \\ |N(x)| & \text{if } s_i = \text{No}\} \end{cases}$$

Construct an exact potential function Φ for this game. Prove that Φ is an exact potential function..

Solution: Define

$$\Phi(x) = -2 \cdot |N(x)|$$

We will show that Φ is an exact potential function which decreases when a player does an improving move.

Consider a strategy profile x_{-i} of all players other than i.

Suppose player i changes their strategy from Yes to No, then the number of countries setting standards goes up by 1 (and thus the potential goes down by 2) which causes the cost of player i to go up by 1. While player i does not have to pay the cost of 3 for setting standards, so net their cost decreases by 2. Hence, the difference in potential value is the same as the difference in cost of player i. More formally, we have argued that,

$$\Phi(\text{No}, x_{-i}) - \Phi(\text{Yes}, x_{-i}) = -2 = c_i(\text{No}, x_{-i}) - c_i(\text{Yes}, x_{-i})$$

The same holds for player i switching from No to Yes. The number of countries not setting standards goes down by 1 (and so the potential goes up by 2), but player i has an extra cost of 3 for setting the standards. The potential decreases by 2 and the cost of player i also decreases by 2. More formally, we have argued that,

$$\Phi(\text{Yes}, x_{-i}) - \Phi(\text{No}, x_{-i}) = 2 = c_i(\text{Yes}, x_{-i}) - c_i(\text{No}, x_{-i})$$

We see that Φ exactly tracks the difference in cost of a player changing strategy and is therefore an exact potential function.

Grading Scheme:

- 12 points
  - 6 points for a correct potential function
  - 6 points for correctly arguing that it is indeed an exact potential function
  - up to 4 points for doing it for 2 players

Question 7

Consider the following mechanism design problem with agents N that are located at the nodes of a binary tree T (i.e. a tree where each node either has two or no children). The mechanism needs to select a "super-node" S. The cost for each agent i that is located at node n_i is the length of the path between S and n_i.

- Start at the root node R. Set $S_0 = R$
- For($k = 0, 1, \ldots$):
  - If S_k has no children, select S_k as the super-node and terminate.
  - Otherwise, let C_1, C_2 be the children of S_k. Ask each agent which of the three nodes S_k, C_1, C_2 it prefers to be selected as the super-node. The agents can vote 'stay' if they prefer S_k, 'left' for C_1 and 'right' for C_2.
  - If there is a clear majority for one of the children C_j (i.e. more agents vote for one of them then for the other two options combined), set $S_{k+1} = C_j$.
  - Otherwise, select S_k as the super-node and terminate.

Assume we are in round l of the algorithm and the option that maximizes the utility of agent $i \in N$ is 'left'. Show that it is a weakly dominant strategy for i to reveal the true preference instead of voting differently. You may assume that all other agents report truthfully in all rounds of the mechanism.

Hint: Think also about what can happen in later rounds.

Solution:

1 If the vote of i does not make a difference for the selection of the algorithm, then it is clearly not beneficial to report wrongly.

2 Assume that the vote of i makes a difference and if i reports truthfully, then $S_{l+1} = C_1$ is selected and the algorithm continues with the next round. In this case, the number of agents that vote for either C_2 and S_l and the number of agents that vote for C_1 are the same (let that number be n).

  - If i reports wrong instead, then either the algorithm ends with $S = S_l$ or continues with $S_{l+1} = C_2$. In both cases, the utility of i is lower than when C_1 is selected and will not increase in a later round.
  - If i reports truthfully, then the algorithm continues with $S_{l+1} = C_1$. The utility for i can never decrease in later rounds, since whenever i votes for staying at the current node in a later round, this vote will receive the majority (as all n agents that voted for C_2 or S_l in round l will vote for staying in later rounds). Therefore, telling the truth in round l is strictly better for i.

Grading Scheme:

- Total: 12 points

- 6 points for arguing that the decision leads to higher utility / lower cost in the case that a misreport would mean a different node being chosen (partial points if sub-arguments are missing / wrong, see individual feedback)
- 6 for analysis that the cost will not increase in later rounds (partial points if sub-arguments are missing / wrong, see individual feedback)

Question 8

Consider the one-to-many Top Trading Cycles (TTC) mechanism with students S and schools C in which each student can be assigned to at most one school and each school $c \in C$ has a capacity q_c of students that can be assigned to it. While we know that TTC is strategyproof for students, the same does not necessarily hold for schools.

Show that it is possible for a school to mis-report its priorities in order to increase the number of students assigned to it. Schools are not able to misreport their capacities, only their priorities.

Solution: It is possible for schools to manipulate in this way. Consider three students and three schools with the following preferences and priorities:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
c_1	c_2	c_2
c_2	c_1	c_3
c_3	c_3	c_1

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
	s_2	s_1	s_2
	s_1	s_3	s_1
	s_3	s_2	s_3
q_c	2	1	1

If the schools give their true priorities, there is a cycle in the first round (s_1, c_1, s_2, c_2). In the second round, s_3 is assigned to c_3. Therefore, the assignments are $(s_1, c_1), (s_2, c2), (s_3, c_3)$.

If school c_1 reports $s_1 \succ s_2 \succ s_3$ instead, the first round only contains the cycle s_1, c_1. Then, in the second round, there is a cycle s_3, c_2 and in the third round a cycle s_2, c_1. In this case, the assignments are $(s_1, c_1), (s_2, c_1), (s_3, c_2)$ and more students are assigned to school c_1.

Grading Scheme:

- Total: 12 points
  - 12 points for the correct analysis of an example in which the courses get at least one additional student. (partial points for errors, see individual feedback)
  - 6 if the example is analysed correctly, but only leads to an improvement of assignments, not of the actual number of students that the course receives (partial points for errors, see individual feedback)