Optimization and Multiagent Systems
Exam

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Question 1

Give an example of a 2-player normal form game that has a pure Nash equilibrium, but also has an initial strategy profile from which the "Best Response Dynamics" algorithm cycles forever and does not terminate.Justify your answer by providing the pure Nash equilibrium strategy profile and the cycle.

Solution:

	A	B	C
A	10, 10	-10,-10	-10,-10
B	-10, -10	1,-1	-1,1
C	-10,-10	-1.1	1,-1

In this game (A, A) is clearly a Nash equilibrium, while $(B, B) \to (B, C) \to (C, C) \to (C, B) \to (B, C)$ is a best response cycle.

Grading Scheme:

- 9 points
  - 3 points for game with Nash equilibrium
  - 4 points for game with cycle
  - 9 points for game with both

Question 2

Consider a normal-form game with two players. Player 1 (the row player) can perform actions A and B, while player 2 (the column player) can perform actions C and D. For $0 < s < 1$ the following bimatrix gives the utilities of the players for the given action profiles:

	C	D
A	0,0	0,-s
B	-s,0	1-s,1-s

Determine all (pure and mixed) Nash equilibria in this game, depending on the parameter c. Justify your answer, showing that there are no other Nash equilibria.

Solution: The pure Nash equilibria are (A,C) and (B,D). There is no other pure Nash equilibrium, since both players would wants to deviate from (A,D) and (B,C) by playing the respective other strategy as that would improve their utility.

The mixed strategy Nash equilibrium can be found as follows. We look for the probability p with which the row player plays A such that the column player is indifferent between playing C and D. This is in the case $0 = -sp + (1-s)(1-p) = -sp + 1 - p - s + sp$ or $p = 1 - s$. Since the game is symmetric, the mixed strategy Nash equilibrium is $(1-s, s), (1-s, s)$. For any other mixed strategy that a player plays, the other player has a pure strategy that is a best response. Therefore, there is no other mixed strategy Nash equilibrium.

Grading Scheme:

- 12 points
  - 3 points per pure Nash equilibrium
  - 6 points for the mixed Nash equilibrium (point deductions for errors in calculations, see individual feedback)

Question 3

You are the organizer of a workshop with different seminars of limited capacity. The students of the workshop have strict and complete preferences over these seminars but can visit only one of them. The seminars have neither preferences nor priorities over the students, i.e., they are indifferent to the students visiting them.

You want to design a mechanism without monetary transfers which is Pareto efficient for the students and where truthful reporting of preference profiles is a (weakly) dominant strategy for the students.

Your assistant proposes the following mechanism: Since you want to allocate the seminar seats as efficiently as possible, you should solve a mixed-integer linear program (MILP). For this, the cost of each student s when assigned to seminar c is defined as the rank that c has in s's preference profile and the objective function is minimizing the sum of costs by assigning the students in such a way that the capacities of the seminars are respected.

Explain, why and in which situation this mechanism does not meet your requirements.

Solution: Truthful reporting is not a dominant strategy for the students. The following offers an example: Assume four courses with capacity 1 and four students with identical preference lists $c_1 \succ c_2 \succ c_3 \succ c_4$. The MILP assigns one student to each course, the sum of costs is always the same (10). Then, either the LP breaks the ties deterministically, or randomly. In the latter case, each student gets an expected rank of 2.5. By submitting a preference list of $c_2 \succ c_1 \succ c_3 \succ c_4$, the MILP will grant this student a place in c_2 to obtain a sum of cost of 9. If the MILP breaks deterministically, this preference list can be submitted by a student getting a course c_3 or c_4 under the initial report. If the ties are broken randomly, the student improves her expected rank. In both cases, this student improves her (expected) rank by reporting non-truthfully.

Grading Scheme:

- 12 points
  - 3 points for determining that the mechanism is manipulable and not strategyproof.
  - 9 points for supplying a proof or precise description in which the manipulation is profitable for a student (point deductions if proof is too vague or contains errors, see individual feedback)

Question 4

Consider the following 2 player bimatrix game where both the row and the column player have three strategies A, B and C.

	A	B	C
A	-1, -11	25, -3	7, -4
B	47, 36	48, 37	19, 10
C	21, 84	-7, 100	100, 50

Give an ordinal potential function for this game. Give your answer as a table where entry B, C contains the value of the potential for the strategy profile (B, C). You don't have to justify your answer.

Solution: An example of an increasing ordinal potential Φ

Φ	A	B	C
A	0	100	50
B	100	200	75
C	90	95	80

where entry (x, y) in the table is the value of $\Phi(x, y)$. Whenever a player has an improving deviation the potential goes up.

For a decreasing potential function we can put minus signs everywhere.

Grading Scheme:

- 9 points
  - -1 point for every comparison error

Questions 5, 6, 7

Consider an auction with two bidders. Every bidder $i = \{1, 2\}$ privately observes his valuation for the object drawn from a uniform distribution over $[0, 1]$, which is common knowledge among players. Assume that all bidders are risk neutral.

The auction is held as a sequential-move auction with the following time structure: Bidder 1 submits his bid $\hat{v}_1$, bidder 2 observes b_1 and responds either buying the item at that price or quits the auction (and then bidder 1 purchases the item for the price b_1)

5. Assume that bidder 2 is a rational agent that wants to maximize his utility. What bid b_1 should bidder 1 report given that his true valuation is v_1?

6. What is the expected revenue of this auction?

7. Is this auction efficient? (If yes, give a formal argument. If not, give a counter-example)

Solution:

5. Bidder 2 will buy the item whenever his valuation is higher than the bid b_1. Therefore, bidder 1's expected utility is given as $\int_0^{b_1}(v_1 - b_1)dv_2 = v_1 b_1 - b_1^2$. Maximizing the expected utility by taking the derivative leads to $b_1 = \frac{1}{2}v_1$ as the optimal bid for bidder 1.

6. Since the valuation of the first bidder is uniformly distributed over $[0, 1]$ and its optimal bid is $\frac{1}{2}v_i$, the expected revenue is $\frac{1}{4}$.

7. The auction is not efficient since the object could go to bidder 2 even if he has a lower valuation for the object, e.g. if bidder 1 has a value of 1 and bidder 2 has a value of 0.6.

Grading Scheme:

5. - 6 points

- 2 points for deriving $\frac{1}{2}v_1$.
- 4 points for correct proof (point deductions for errors in the proof, see individual feedback)

6. • 3 points
   - 3 points for deriving $\frac{1}{4}$
   - (up to 2 points for partial argument that does not result in $\frac{1}{4}$ but is correct up to that)

7. • 3 points
   - 3 points for correct answer and counter example.

Questions 8, 9

Consider the following mechanism for a combinatorial auction setting with quasilinear utilities of the participants:

- Bidders submit bids on bundles of items.
- We use an XOR bidding.
- The mechanism then calculates the efficient allocation of items (maximizing the utilitarian social welfare) based on the received bids, assigning at most one bundle to each bidder.
- If bidder i receives S, the bidder i has to pay as follows:
  - If there is a subset $S' \subseteq S$ for which another bidder $j \neq i$ submitted a bid, bidder i has to pay $\max_{j \neq i, S' \subseteq S} b_j(S')$
  - If no such subset S' exists: If there is a set T with $T \cap S \neq \emptyset$ for which another bidder $j \neq i$ submitted a bid, bidder i has to pay $\max_{j \neq i, T \cap S \neq \emptyset} b_j(T)$
  - If none of these sets exists, the bidder i pays nothing.

8. Either proof that reporting true valuations is a (weakly) dominant strategy for each bidder or show with the help of an example that it may be beneficial for a bidder to report false valuations.
9. Either proof that using this mechanism, bidders can never make a loss when reporting their true valuations (i.e. bidder's payment never exceed their bid) or show with the help of an example that bidders can make a loss.

Solution:

8. There are several examples that show that bidders can benefit from reporting wrong valuations, for example: 3 bidders and 3 items A, B, C and the following true valuations:

   - $v_1(AB) = 10$
   - $v_2(BC) = 20$
   - $v_3(A) = 9$

   In this case, bidder 1 receives no items and has a utility of 0. Bidder 1 can instead submit a bid of 30 and will be assigned the bundle $\{A, B\}$, pay 9 and have a utility of 1. Therefore, truthful reporting is not a (weakly) dominant strategy in the proposed mechanism.

9. Again, there are several examples that show that bidders can make a loss with this mechanism, for example: 3 bidders and 3 items A, B, C and the following reported (true) valuations:

   - $v_1(AB) = 10$
   - $v_2(BC) = 20$
   - $v_3(C) = 15$

   The mechanism assigns $\{A, B\}$ to bidder 1 and $\{C\}$ to bidder 3. Bidder 1 would need to pay 20 and have a utility of -10. Therefore, bidders can make a loss when reporting their true values to the proposed mechanism.

Grading Scheme:

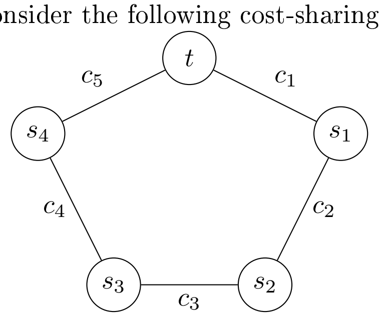
8. - 10 points
     - If providing an example or proof that the mechanism is not truthful: Point reductions for errors in the example, or, in the case of a proof with abstract examples, for missing statements or wrong assumptions (depending on severity of error, see individual feedback)
     - Up to 4 points if formal proof tried for showing that the mechanism is truthful and the proof leads to the wrong conclusion because of missing some arguments / wrong assumptions. (Less points if proof contains many errors / wrong assumptions or makes no sense, see individual feedback)

9. - 10 points
     - If providing an example or proof that the mechanism can lead to a loss for bidders: Point reductions for errors in the example, or, in the case of a proof with abstract examples, for missing statements or wrong assumptions (depending on severity of error, see individual feedback)

– Up to 4 points if formal proof tried for showing that the mechanism is truthful and the proof leads to the wrong conclusion because of missing some arguments / wrong assumptions. (Less points if proof contains many errors / wrong assumptions or makes no sense, see individual feedback)

Questions 10, 11

Consider the following cost-sharing game.



The game has 4 players, and every player i wants to be connected from s_i to t. Every edge e has a positive cost $c_e > 0$ and can be travelled in both directions. When a player uses an edge e alone they pay the full price c_e, but when two or more players use the same edge they will share the cost c_e equally.

The social cost is the sum of the costs of the players. We already know that this game admits an exact potential function and therefore always has a Nash equilibrium.

10. Can you give a social optimum, i.e., a strategy that minimizes the social cost?

11. Show that the price of stability is at most 2. (Hint: you don't have to use an argument involving a potential function)

Solution:

10. Note that the social cost is equal to summing the cost of all used edges. A strategy minimizing the social cost will leave out the most expensive edge and route all the players over the other edges.

11. Suppose the social optimum, with cost OPT, is not a Nash equilibrium, otherwise the price of stability is 1. Then some player j, who currently incurs some cost C_j has a profitable deviation. Let e^* be the most expensive edge. Since a change in strategy for player j always involves e^* we know that $c_{e^*} < C_j \leq OPT$. OPT is the sum of the cost of alle edges except for c_e^*, so then the sum of all edges is at most $OPT + c_{e^*} < 2 \cdot OPT$. If all edges sum up to at most $2 \cdot OPT$ then a Nash equilibrium (which we know exists) can then also have cost at most $2 \cdot OPT$.

Grading Scheme:

10. - 6 points
        - 3 points for realization that any feasible solution leaves out at most one edge
        - 3 points for noticing that we can take out the most expensive edge
        - 2 points if reasoned for $c_1 = c_2 = c_3 = c_4 = c_5 = 1$.

11. - 10 points
        - 2 points for case distinction OPT is Nash vs OPT is not Nash
        - 3 points for realizing that an improving move involves the most expensive edge
        - 2 points for bounding the cost of the most expensive edge by OPT
        - 3 points for showing that any Nash equilibrium then has cost at most $2 \cdot OPT$.

    alternative: 3 points for trying to invoke bounds for affine congestion games (although this is not an affine congestion game)