Write your names and student numbers on each sheet that you hand in. Your answer should give a clear explanation, with intermediate steps and calculations included. Good luck!

1. Consider the following LP which is referred to as the "primal LP".

$$\begin{aligned} \min w &= 2x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & 2x_1 + 2x_2 + x_3 \geq 13 \\ & -x_1 + 2x_2 + 4x_3 = 7 \\ & x_1 \text{ unrestricted}, x_2 \geq 0, x_3 \leq 0 \end{aligned}$$

   (a) Write down the dual LP of the above primal LP.

   (b) The above primal LP has been rewritten to an LP with only non-negative decision variables $x_1', x_1'', x_2, x_3' \geq 0$ by substituting $x_1 = x_1' - x_1''$ and $x_3' = -x_3$. Subsequently the rewritten LP is transformed to standard form for the simplex method by introducing surplus variable s_1 in the first constraint, artificial variable r_1 in the first constraint and artificial variable r_2 in the second constraint. Next the problem has been solved by applying the two phase simplex method. The final simplex tableau (the tableau obtained at the end of the second phase) is as follows:

Basic	w	x_1'	x_1''	x_2	x_3'	s_1	r_1	r_2	value
w	1	0	0	0	-1	$-\frac{4}{3}$	$\frac{4}{3}$	$\frac{2}{3}$	22
x_1'	0	1	-1	0	1	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	2
x_2	0	0	0	1	$-\frac{3}{2}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{9}{2}$

   Briefly explain how you can see that the solution cooresponding to the above simplex tableau is optimal. Next give the optimal solution for the original decision variables x_1, x_2, x_3 of the primal LP and give the optimal objective value w^*. Also write down the optimal solution of the dual LP and optimal dual objective value.

   (c) Suppose the right hand sides 13 and 7 of the two contraints of the primal LP are changed to respectively 14 and 5. Determine the new optimal solution for variables x_1, x_2, x_3 and new optimal objective value w^*.

   (d) Instead of changing the right hand sides of the constraints we now add the extra constraint $x_1 \leq 1$ to the original primal LP. Find an optimal solution of this modified LP by applying the dual simplex method.

   Instruction: Do not start from scratch but make use of the final simplex tableau for the original primal LP which was given in exercise 1b. Since the modified LP is feasible the two artificial variables should never re-enter the basis. Therefore to save time you can omit the columns of the two artificial variables from the simplex tableau before doing the dual simplex iteration.

Solution:

(a) Using the "dictionary" method, we obtain

$$\begin{aligned} \max \quad & z = 13y_1 + 7y_2 \\ \text{s.t.} \quad & 2y_1 - y_2 = 2 \\ & 2y_1 + 2y_2 \leq 4 \\ & y_1 + 4y_2 \geq 3 \\ & y_1 \geq 0 \\ & y_2 \text{ unrestricted} \end{aligned}$$

(b) The optimal solution of the primal LP is $x_1^* = 2$, $x_2^* = \frac{9}{2}$, $x_3^* = 0$ and optimal objective value $w^* = 22$.
The optimal solution of the dual LP is $y_1^* = \frac{4}{3}$ (the coefficient of r_1 in the w-row), $y_2^* = \frac{2}{3}$ (the coefficient of r_2 in the w-row) and $z^* = 22$ (confirming the strong duality theorem).

(c) The change in right hand side of the first constraint is $\Delta_1 = 14 - 13 = 1$ and in right hand side of the second constraint is $\Delta_2 = 5 - 7 = -2$.
Hence the value in the w-row becomes $22 + \frac{4}{3}\Delta_1 + \frac{2}{3}\Delta_2 = 22$, the value in the x_1'-row becomes $2 + \frac{1}{3}\Delta_1 - \frac{1}{3}\Delta_2 = 3$ and the value in the x_2-row becomes $\frac{9}{2} + \frac{1}{6}\Delta_1 + \frac{1}{3}\Delta_2 = 4$. Since the values of basic variables x_1' and x_2 are still nonnegative the corresponding solution is feasible and optimal.
Hence the optimal solution for this LP is $x_1^* = 3$, $x_2^* = 4$, $x_3^* = 0$ and $w^* = 22$.

(d) We rewrite the new constraint $x_1 \leq 1$ to equality $x_1 + s_3 = 1$ in which variables x_1, s_3 are nonnegative and s_3 is a new slack variable. From the simplex tableau given in 1b we omit the columns for artificial variables r_1 and r_2, add a column for the new slack variable s_3 and we add an extra row for the new constraint. This gives the following tableau.

Basic	w	x_1'	x_1''	x_2	x_3'	s_1	s_3	value
w	1	0	0	0	-1	$-\frac{4}{3}$	0	22
x_1'	0	1	-1	0	1	$-\frac{1}{3}$	0	2
x_2	0	0	0	1	$-\frac{3}{2}$	$-\frac{1}{6}$	0	$\frac{9}{2}$
s_3	0	1	0	0	0	0	1	1

To restore the column of basic variable x_1' subtract the x_1'-row from the s_3-row resulting in tableau:

Basic	w	x_1'	x_1''	x_2	x_3'	s_1	s_3	value
w	1	0	0	0	-1	$-\frac{4}{3}$	0	22
x_1'	0	1	-1	0	1	$-\frac{1}{3}$	0	2
x_2	0	0	0	1	$-\frac{3}{2}$	$-\frac{1}{6}$	0	$\frac{9}{2}$
s_3	0	0	1	0	-1	$\frac{1}{3}$	1	-1

Since the s_3 value is now negative it follows (applying the dual simplex method) that s_3 has to leave the basis. Since x_3' is the only variable with negative cofficient in the s_3-row it follows that x_3' enters the basis. Applying the corresponding pivot step the following tableau is obtained.

Basic	w	x_1'	x_1''	x_2	x_3'	s_1	s_3	value
w	1	0	-1	0	0	$-\frac{5}{3}$	-1	23
x_1'	0	1	0	0	0	0	-1	1
x_2	0	0	$-\frac{3}{2}$	1	0	$-\frac{2}{3}$	$-\frac{3}{2}$	6
x_3'	0	0	-1	0	1	$-\frac{1}{3}$	-1	1

From this tableau it follows that the optimal solution of this LP is:
$x_1^* = 1$, $x_2^* = 6$, $x_3^* = -1$, $w^* = 23$.

2. A maintenance company has to do three maintenance tasks ($i = 1, 2, 3$) for which five employees ($j = 1, 2, 3, 4, 5$) are available. Each employee can be assigned to at most one task. For each task one employee is needed except task number 1 for which two employees are needed to complete the task. Task number 3 can not be done by employees 2 and 5. Besides that restriction each employee is able to do each task. The distances in kilometers between the location of task i and the home of employee j is d_{ij} and all these distances d_{ij} are known. The objective of the maintenance company is to assign employees to the tasks such that all the tasks will be completed and the total travel distance is minimized. Formulate the problem of assigning employees to the tasks minimizing the total travel distance as an integer linear program (ILP).

Solution:

Define as decision variables for $i = 1, 2, 3,\ j = 1, 2, 3, 4, 5$ the binary variables x_{ij} where $x_{ij} = 1$ if employee j is assigned to task i. Then the total travel distance is $\sum_{i=1}^{3}\sum_{j=1}^{5} d_{ij}x_{ij}$ (this is the total distance for the employees travelling to the locations. If you also want to consider the travel distance for them to return to their homes you could multiply the above objective function by a factor 2. Multiplying the objective function with some positive number will not change the optimal solution, only the corresponding optimal value will be multiplied by that number).
To make sure that each employee is assigned to at most one task we have for each employee j the constraint $\sum_{i=1}^{3} x_{ij} \leq 1$. To make sure that exactly 2 employees are assigned to task 1 we have the constraint $\sum_{j=1}^{5} x_{1j} = 2$. Similarly for $i = 2, 3$ we have $\sum_{j=1}^{5} x_{ij} = 1$. Finally to make sure that employees 2 and 5 are not assigned to task 3 we add the constraint $x_{32} + x_{35} = 0$. Combining the objective function and the constraints for the problem we obtain the following ILP formulation of the problem.

$$\text{minimize } w = \sum_{i=1}^{3}\sum_{j=1}^{5} d_{ij}x_{ij}$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{i=1}^{3} x_{ij} \leq 1 \text{ for } j = 1, 2, 3, 4, 5 \\
& \sum_{j=1}^{5} x_{1j} = 2 \\
& \sum_{j=1}^{5} x_{ij} = 1 \text{ for } i = 2, 3 \\
& x_{32} + x_{35} = 0 \\
& \text{all } x_{ij} \in \{0, 1\} \text{(binary decision variables)}
\end{aligned}$$