
Write your names and student numbers on each sheet that you hand in. Your answer should give a clear explanation, with intermediate steps and calculations included. Good luck!

1. A company is producing three products (A, B, and C) in the upcoming period. These three products are crafted from the same two raw materials (1 and 2).

To craft one unit of product A, 10 kg of raw material 1 and 30 kg of raw material 2 are needed.

To craft one unit of product B, 40 kg of raw material 1 and 10 kg of raw material 2 are needed.

To craft one unit of product C, 20 kg of raw material 1 and 30 kg of raw material 2 are needed.

Maximal 50 units of product A can be sold for a price of 300 euros per unit. Maximal 40 units of product B can be sold for a price of 550 euros per unit. Maximal 45 units of product C can be sold for a price of 500 euros per unit.

There is a regulation that at least 40 % of the produced units should be of product A (the cheapest of the three products). The current purchase prices for raw materials are 11 euros per kg for raw material 1 and 8 euros per kg for raw material 2. There is no limit on the amounts of raw material the company can purchase for these standalone prices. Additionally, the company has an option to purchase raw material 1 and raw material 2 together for the same amount of kg. Such a bundle containing equal weights of both raw materials has a 10% discount in price compared to purchasing raw materials 1 and 2 separately for the current standalone prices. The 10% discount in price is offered for bundles which contain at most 1000 kg of each of the two raw materials.

The company needs to create a production plan for the upcoming period. The objective of the company is to maximize the net profit in the upcoming period through crafting and selling of the three products given the current prices and satisfying the constraints following from the problem description.

Formulate a linear program (LP) that describes this optimization problem maximizing the total netto profit for the company. Explain the meaning of all decision variables and constraints of your LP formulation. Be careful to (re)write all constraints as a linear (in)equality. Be clear if you make any assumptions or simplifications in your LP formulation.

You do **not** need to solve the LP you have formulated! The most challenging part of this problem is figuring out what is a good choice of decision variables. Think about this carefully.

Solution exercise 1:

Decision variables: define decision variables x_A, x_B, x_C as the number of units of the product cooresponding with the index which are crafted.

Define for $j = 1, 2$ decison variable y_j as the amount of kg which is purchased of raw material j for the standalone price. Define decison variable y_3 as the amount of kg which is purchased as bundle of both materials for the discounted price. Notice that for such a bundle the price per kg of both raw materials will be $0.90(11+8) = 17.1$ euro.

Objective function: $z = 300x_A + 550x_B + 500x_C - 11y_1 - 8y_2 - 17.1y_3$ is the total netto profit for the company which should be maximized.

Constraints: $x_a \leq 50, x_B \leq 40, x_C \leq 45$ the selling constraints. $y_1 + y_3 \geq 10x_A + 40x_B + 20x_C$, $y_2 + y_3 \geq 30x_A + 10x_B + 30x_C$ the raw material constraints. $y_3 \leq 1000$ (the discount constrained) Moreover, the production regulation constraint implies $\frac{x_A}{x_A + x_B + x_C} \geq 0.40$ which should be rewritten to the linear inequality $0.60x_A \geq 0.40x_B + 0.40x_C$.

Moreover, we also have nonnegativity constraints for all decision variables.

This gives the following LP formulation:

$$\begin{array}{llll}
 \max & z & = & 300x_A + 550x_B + 500x_C - 11y_1 - 8y_2 - 17.1y_3 \\
 \text{subject to:} & x_A & \leq & 50 \\
 & x_B & \leq & 40 \\
 & x_C & \leq & 45 \\
 & y_1 + y_3 & \geq & 10x_A + 40x_B + 20x_C \\
 & y_2 + y_3 & \geq & 30x_A + 10x_A + 30x_A \\
 & y_3 & \leq & 1000 \\
 & 0.60x_A & \geq & 0.40x_B + 0.40x_C \\
 & x_A, x_B, x_C, y_1, y_2, y_3 & \geq & 0
 \end{array}$$

2. Solve the following LP using the simplex method. In the first pivot step choose x_1 as basis entering variable. If you then apply the simplex method correct you will need two pivot steps to find the optimal solution.

$$\begin{array}{llll}
 \max & z = 5x_1 + 2x_2 - x_3 & & \\
 \text{subject to} & 4x_1 + 2x_2 + x_3 & \leq & 17 \\
 & 2x_1 + x_2 - 2x_3 & \leq & 6 \\
 & x_1, x_2, x_3 \geq 0 & &
 \end{array}$$

Solution exercise 2:

The first simplex tableau is:

basic	z	x_1	x_2	x_3	s_1	s_2	value
z	1	-5	-2	1	0	0	0
s_1	0	4	2	1	1	0	17
s_2	0	2	1	-2	0	1	6

Choose x_1 as entering variable. Because $\min(\frac{17}{4}, \frac{6}{2}) = 3$ it follows that s_2 will leave the basis. Then the next simplex tableau becomes:

basic	z	x_1	x_2	x_3	s_1	s_2	value
z	1	0	0.50	-4	0	2.50	15
s_1	0	0	0	5	1	-2	5
x_1	0	1	0.50	-1	0	0.50	3

Now x_3 is the only variable with negative coefficient in the z -row and thus x_3 enters the basis. Then it follows that s_1 will leave the basis. The next simplex tableau becomes:

basic	z	x_1	x_2	x_3	s_1	s_2	value
z	1	0	0.50	0	0.80	0.90	19
x_3	0	0	0	1	0.20	-0.40	1
x_1	0	1	0.50	0	0.20	0.10	4

Hence an optimal solution is: $x_1^* = 4, x_2^* = 0, x_3^* = 1$ and $z^* = 19$.