Exam Operations Research

Date: June 8, 2022
Time: 18:45 - 21:00

Points per exercise:

- Exercise 1 has a total of 30 points (a(10), b(5), c(5) and d(10)).
- Exercise 2 has a total of 15 points.
- Exercise 3 has a total of 15 points (a(5) and b(10)).
- Exercise 4 has a total of 10 points.
- Exercise 5 has a total of 20 points (a(10) and b(10)).

Thus in total 90 points can be obtained. The exam grade is determined as follows:

$$\text{Exam grade } = 1 + \frac{\text{total number of obtained points}}{10}.$$

- Calculator is allowed.
- This exam consists of 6 pages, including this one.
- The duration of this exam is **2 hours and 15 minutes**.
- Students who have obtained permission for extra time may use an *additional 30 minutes*.

Exercise 1

Consider the following LP which is referred to as the "primal LP".

$$\begin{aligned} \min \quad & w = 4x_1 - x_2 + x_3 + 2x_4 \\ \text{s.t.} \quad & 3x_1 - 2x_2 + x_3 + x_4 \geq 18 \\ & x_1 + x_2 + 2x_3 + 2x_4 = 16 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(a) [**10 points**] Determine the dual of this LP.

(b) [**5 points**] The primal LP has been solved by the simplex method resulting in the following final tableau where s_1 is the surplus variable introduced in the first constraint, r_1 is the artificial variable introduced in the first constraint and r_2 is the artificial variable introduced in the second constraint.

Basic	w	x_1	x_2	x_3	x_4	s_1	r_1	r_2	value
w	1	0	-2	0	-1	-1.40	1.40	-0.20	22
x_1	0	1	-1	0	0	-0.40	0.40	-0.20	4
x_3	0	0	1	1	1	0.20	-0.20	0.60	6

Determine the optimal solution of the primal LP and optimal objective value. Determine also the optimal solution of the dual LP and optimal dual objective value.

(c) [**5 points**] Suppose the right hand sides of the first constraint is changed to 25 (instead of 18) and the right hand side of the second constraint is changed to 20 (intsead of 16). Determine the optimal solution and optimal objective value of this modified LP.

(d) [**10 points**] Now assume the right hand side of the first constraint is changed to 8 (instead of 18) and the right hand side of the second constraint is changed to 21 (instead of 16). Apply the dual simplex method to determine the optimal solution and optimal objective value of this modified LP.
Instruction: Since artificial variables should never become basic variables when applying the dual simplex method the columns of the two artificial variables can be omitted from the simplex tableau when you have chosen the pivot row. Moreover, it will require only one pivot step to find the new optimal solution.

Exercise 2

Three friends go on a journey. There are n items they need to take with them on the journey and the weight of item i is w_i for $i = 1, 2, \dots, n$. On the journey each of them will carry one bag and the n items have to be divided over the 3 bags. They want to divide the items over the bags such that the weight of the heaviest bag is as small as possible. In other words they want to divide the n items over the 3 bags in a fair way such that it will not happen that one of the bags is much heavier than the other two bags.

(a) **[15 points]** Formulate the objective of the three friends as an integer linear program (ILP). Explain all variables and constraints in your ILP formulation of this problem.
Instruction: You may impose in your formulation (since it does not change the optimal solution) that bag 1 gets at least as much total weight as bag 2 and bag 2 gets at least as much total weight as bag 3. Using these restrictions in the formulation will make it easier to formulate the problem as an ILP.

Exercise 3

Consider the following ILP with four binary decision variables x_1, x_2, x_3, x_4.

$$\begin{aligned} \max \quad & z = 17x_1 + 20x_2 + 14x_3 + 11x_4 \\ \text{s.t.} \quad & 8x_1 + 9x_2 + 7x_3 + 5x_4 \leq 23 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

(a) [**5 points**] Write down the LP relaxation of this ILP and determine the unique optimal solution of the LP relaxation. Also determine the corresponding optimal objective value of the LP relaxation.
Instruction: Since the ILP is a $0 - 1$ *knapsack problem it is possible to quickly determine the optimal solution of the LP relaxation. You do not need to use the simplex method for that.*

(b) [**10 points**] Solve the ILP by applying the **branch-and-bound** method. Clearly indicate in which order you compute the nodes of the search tree and where you prune the search tree. Also indicate which pruning criterion you use when you prune.

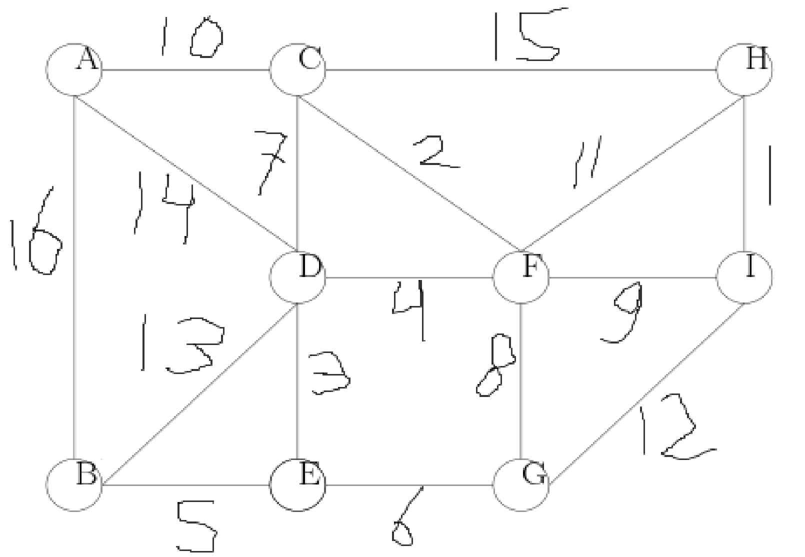

Figure 1: Exercise 4

Exercise 4

(a) **[10 points]** Consider the problem of finding a minimum weight spanning tree in the non-directed graph of Figure 1 (where edge weights have been indicated in the figure) using Kruskal's algorithm. Make clear in which **order** the edges are picked by the algorithm and draw the minimum weight spanning tree which is finally obtained. Also explicitly write down the instructions of Kruskal's algorithm which prescribe the order in which edges to be included in the minimum spanning tree are picked.

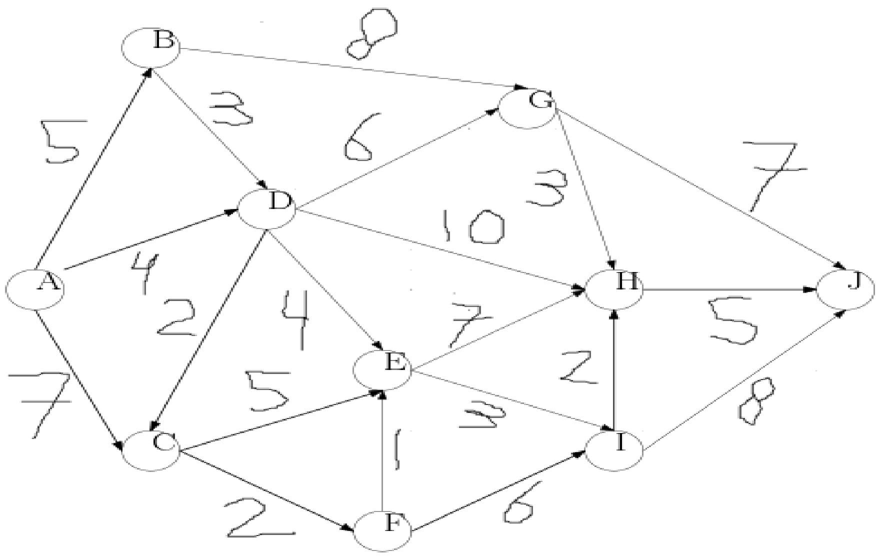

Figure 2: Exercise 5

Exercise 5

Consider the directed acyclic graph as drawn in Figure 2 with lengths of the arcs as indicated in the figure.

(a) **[10 points]** Apply Dijkstra's algorithm to determine the **shortest path** from node A to node J in this acyclic directed graph. It should be clear from the computations you write down that you have applied Dijkstra's algorithm to find the shortest path. After finishing the computations also indicate explicitly the shortest path which is obtained and the length of that shortest path.

(b) **[10 points]** Apply backward recursion to determine the **longest path** from node A to node J in the acyclic directed graph. Make sure it is clear in which order you calculate the function values and explain why the function values are calculated in that order.
Besides the function values calculated by applying backward recursion also indicate explicitly the longest path which is obtained and the length of that longest path.