Exam Operations Research

Date: June 8, 2022
Time: 18:45 - 21:00

Points per exercise:

- Exercise 1 has a total of 30 points (a(10), b(5), c(5) and d(10)).
- Exercise 2 has a total of 15 points.
- Exercise 3 has a total of 15 points (a(5) and b(10)).
- Exercise 4 has a total of 10 points.
- Exercise 5 has a total of 20 points (a(10) and b(10)).

Thus in total 90 points can be obtained. The exam grade is determined as follows:

$$\text{Exam grade } = 1 + \frac{\text{total number of obtained points}}{10}.$$

- Calculator is allowed.
- This exam consists of 6 pages, including this one.
- The duration of this exam is **2 hours and 15 minutes**.
- Students who have obtained permission for extra time may use an *additional 30 minutes*.

Exercise 1

Consider the following LP which is referred to as the "primal LP".

$$
\begin{aligned}
\min \quad & w = 4x_1 - x_2 + x_3 + 2x_4 \\
\text{s.t.} \quad & 3x_1 - 2x_2 + x_3 + x_4 \geq 18 \\
& x_1 + x_2 + 2x_3 + 2x_4 = 16 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

(a) **[10 points]** Determine the dual of this LP.

(b) **[5 points]** The primal LP has been solved by the simplex method resulting in the following final tableau where s_1 is the surplus variable introduced in the first constraint, r_1 is the artificial variable introduced in the first constraint and r_2 is the artificial variable introduced in the second constraint.

Basic	w	x_1	x_2	x_3	x_4	s_1	r_1	r_2	value
w	1	0	-2	0	-1	-1.40	1.40	-0.20	22
x_1	0	1	-1	0	0	-0.40	0.40	-0.20	4
x_3	0	0	1	1	1	0.20	-0.20	0.60	6

Determine the optimal solution of the primal LP and optimal objective value. Determine also the optimal solution of the dual LP and optimal dual objective value.

(c) **[5 points]** Suppose the right hand sides of the first constraint is changed to 25 (instead of 18) and the right hand side of the second constraint is changed to 20 (instead of 16). Determine the optimal solution and optimal objective value of this modified LP.

(d) **[10 points]** Now assume the right hand side of the first constraint is changed to 8 (instead of 18) and the right hand side of the second constraint is changed to 21 (instead of 16). Apply the dual simplex method to determine the optimal solution and optimal objective value of this modified LP.
Instruction: Since artificial variables should never become basic variables when applying the dual simplex method the columns of the two artificial variables can be omitted from the simplex tableau when you have chosen the pivot row. Moreover, it will require only one pivot step to find the new optimal solution.

Solution exercise 1:

(a) The dual LP is:

$$\begin{aligned} \max \quad & z = 18y_1 + 16y_2 \\ \text{s.t.} \quad & 3y_1 + y_2 \leq 4 \\ & -2y_1 + y_2 \leq -1 \\ & y_1 + 2y_2 \leq 1 \\ & y_1 + 2y_2 \leq 2 \\ & y_1 \geq 0, y_2 \text{ unrestricted} \end{aligned}$$

(b) The optimal solution of the primal LP is $x_1^* = 4$, $x_2^* = 0$, $x_3^* = 6$, $x_4^* = 0$ and optimal objective value $w^* = 22$. The optimal solution of the dual LP is $y_1^* = 1.40$, $y_2^* = -0.20$ with optimal dual objective value $z^* = 22$.

(c) The change in the right hand side of the first constraint is $\Delta_1 = 25 - 18 = 7$ and in the second constraint the change is $\Delta_2 = 20 - 16 = 4$. Thus as corresponding basic solution of the modified LP we obtain $x_1^* = 4 + 0.40\Delta_1 - 0.20\Delta_2 = 6$ and $x_3^* = 6 - 0.20\Delta_1 + 0.60\Delta_2 = 7$ and for the nonbasic variables we still have $x_2^* = x_4^* = 0$. Notice that this solution is feasible and thus optimal for the modified LP. It follows that the new optimal objective value is $w^* = 22 + 1.40\Delta_1 - 0.20\Delta_2 = 31$.

(d) Now the change in right hand side of the first constraint is $\Delta_1 = 8 - 18 = -10$ and in the second constraint the change is $\Delta_2 = 21 - 16 = 5$. Computing the corresponding new solution similar as in 1c we obtain $x_1 = 4 + 0.40\Delta_1 - 0.20\Delta_2 = -1$ and $x_3 = 6 - 0.20\Delta_1 + 0.60\Delta_2 = 11$. For the nonbasic variables we still have $x_2 = x_4 = 0$ and the new objective value is $w = 22 + 1.40\Delta_1 - 0.20\Delta_2 = 7$. However, this solution is not feasible (and thus also not optimal) since x_1 is negative and therefore the dual simplex method has to be applied.

Omitting the columns of the two artificial variables and filling in the values calculated above in the value column gives the following simplex tableau:

Basic	w	x_1	x_2	x_3	x_4	s_1	value
w	1	0	-2	0	-1	-1.40	7
x_1	0	1	-1	0	0	-0.40	-1
x_3	0	0	1	1	1	0.20	11

Since x_1 is negative and has to leave the basis it follows that x_2 will enter the basis by the minimal ratio rule since $|\frac{-2}{-1}| < |\frac{-1.40}{-0.40}|$. We multiply the row of new basis variable x_2 by -1 to get a 1 at the pivot position. Then we have the following simplex tableau:

Basic	w	x_1	x_2	x_3	x_4	s_1	value
w	1	0	-2	0	-1	-1.40	7
x_2	0	-1	1	0	0	0.40	1
x_3	0	0	1	1	1	0.20	11

Now the column of new basis variable x_2 has still to be transformed to unit vector. Therefore 2 times x_2-row is added to the w-row and x_2-row is subtracted from the x_3-row. We obtain the following tableau:

Basic	w	x_1	x_2	x_3	x_4	s_1	value
w	1	-2	0	0	-1	-0.60	9
x_2	0	-1	1	0	0	0.40	1
x_3	0	1	0	1	1	-0.20	10

From this tableau it can be concluded that the optimal solution of the modified LP is $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 10$, $x_4^* = 0$ with optimal objective value $w^* = 9$.

Exercise 2

Three friends go on a journey. There are n items they need to take with them on the journey and the weight of item i is w_i for $i = 1, 2, \ldots, n$. On the journey each of them will carry one bag and the n items have to be divided over the 3 bags. They want to divide the items over the bags such that the weight of the heaviest bag is as small as possible. In other words they want to divide the n items over the 3 bags in a fair way such that it will not happen that one of the bags is much heavier than the other two bags.

(a) **[15 points]** Formulate the objective of the three friends as an integer linear program (ILP). Explain all variables and constraints in your ILP formulation of this problem.
Instruction: You may impose in your formulation (since it does not change the optimal solution) that bag 1 gets at least as much total weight as bag 2 and bag 2 gets at least as much total weight as bag 3. Using these restrictions in the formulation will make it easier to formulate the problem as an ILP.

Solution exercise 2: Define binary variables x_{ij} for $i = 1, 2, \ldots, n$ and $j = 1, 2, 3$ where $x_{ij} = 1$ if item i is put in bag j and $x_{ij} = 0$ if item i is not put in bag j Also define auxiliary decision variables y_j for $j = 1, 2, 3$ where y_j will be the total weight which is put in bag j. Using these decision variables the problem can be formulated as the following ILP:

$$
\begin{aligned}
\min \quad & w = y_1 \\
\text{s.t.} \quad & y_j = \sum_{i=1}^{n} w_i x_{ij} \text{ for } j = 1, 2, 3 \\
& y_1 \geq y_2 \\
& y_2 \geq y_3 \\
& \sum_{j=1}^{3} x_{ij} = 1 \text{ for } i = 1, 2, \ldots, n \\
& \text{All variables } x_{ij} \in \{0, 1\}
\end{aligned}
$$

The first constraints make sure that auxiliary variable y_j will be the total weight whis is put in bag j. The second and third constraint make sure that y_1 is the maximal weight of the three bags. Minimizing this maximal weight y_1 becomes then the objective of the ILP. The fourth constraints make sure every item i is put in exactly one bag j. Finally there are the binary constraints for the x_{ij} variables. It is allowed to also add nonnegativity constraints for the y_j variables, but that is not necessary since that follow already from contraints 1.

Exercise 3

Consider the following ILP with four binary decision variables x_1, x_2, x_3, x_4.

$$\begin{aligned} \max \quad & z = 17x_1 + 20x_2 + 14x_3 + 11x_4 \\ \text{s.t.} \quad & 8x_1 + 9x_2 + 7x_3 + 5x_4 \leq 23 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

(a) **[5 points]** Write down the LP relaxation of this ILP and determine the unique optimal solution of the LP relaxation. Also determine the corresponding optimal objective value of the LP relaxation.
Instruction: Since the ILP is a $0-1$ knapsack problem it is possible to quickly determine the optimal solution of the LP relaxation. You do not need to use the simplex method for that.

(b) **[10 points]** Solve the ILP by applying the **branch-and-bound** method. Clearly indicate in which order you compute the nodes of the search tree and where you prune the search tree. Also indicate which pruning criterion you use when you prune.

Solution exercise 3:

(a) The ILP has the following LP relaxation:

$$\begin{aligned} \max \quad & z = 17x_1 + 20x_2 + 14x_3 + 11x_4 \\ \text{s.t.} \quad & 8x_1 + 9x_2 + 7x_3 + 5x_4 \leq 23 \\ & 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3, 4 \end{aligned}$$

Since it is an $0-1$ knapsack problem the optimal solution of the LP relaxation can be found by ordering the four variables according to value-weight ratio and then applying the greedy algorithm. Since $\frac{20}{9} > \frac{11}{5} > \frac{17}{8} > \frac{14}{7}$ the greedy algorithm gives as optimal solution of the LP relaxation $x_2 = 1$, $x_4 = 1$, $x_1 = 1$, $x_3 = \frac{1}{7}$ with corresponding optimal objective value $z^* = 50$.

(b) From (a) we already have for starting node 0 the upperbound $z^0 = 50$ and we have to branch on the fractional variable x_3.
Put $x_3 = 0$ for subproblem 1 (corresponding to node 1 in the search tree). Then the LP relaxation has optimal solution $x_3 = 0$, $x_2 = 1$, $x_4 = 1$, $x_1 = 1$ with value $z^1 = 48$. This solution is feasible for the original ILP and thus the objective value 48 is a lower bound for the original ILP.
Put $x_3 = 1$ for subproblem 2 (corresponding to node 2 in the search tree). The LP relaxation in node 2 has optimal solution $x_3 = 1$, $x_2 = 1$, $x_4 = 1$, $x_1 = \frac{2}{8} = \frac{1}{4}$ with value $z^2 = 49\frac{1}{4}$. Branch on variable x_1 from subproblem 2. Putting $x_3 = 1$, $x_1 = 0$ for subproblem 2.1 gives solution $x_3 = 1$, $x_1 = 0$, $x_2 = 1$, $x_4 = 1$ with $z^{2.1} = 45$ which is less than lowerbound 48. Putting $x_3 = 1$, $x_1 = 1$ for subproblem 2.2 gives solution $x_3 = 1$, $x_1 = 1$, $x_2 = \frac{8}{9}$, $x_4 = 0$ with $z^{2.2} = 48\frac{7}{9}$. However since any feasible ILP solution has integer obejective value it now follows that an upper bound for feasible solutions in subproblem 2.2 is $\lfloor z^{2.2} \rfloor = 48$ which is not above the lowerbound. Hence we can prune and conclude that $x_2 = 1$, $x_4 = 1$, $x_1 = 1$, $x_3 = 0$ is an optimal solution of the original ILP and the optimal objective value of the ILP is $z^* = 48$.

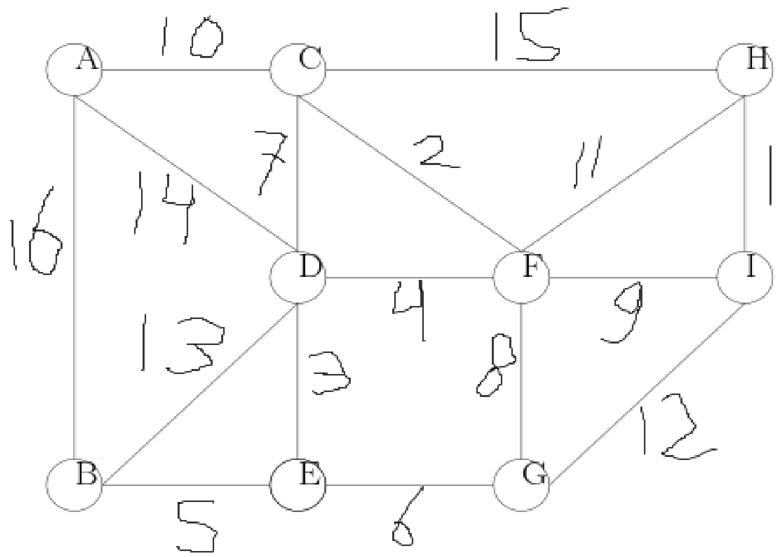
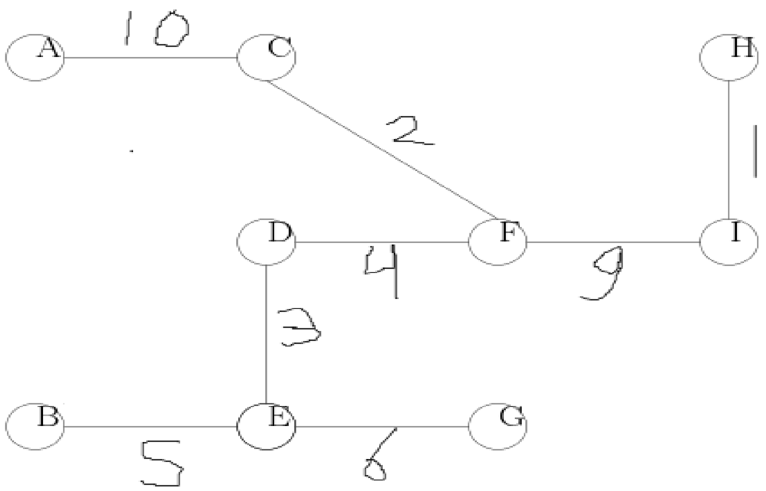


Figure 1: Exercise 4

Exercise 4

(a) **[10 points]** Consider the problem of finding a minimum weight spanning tree in the non-directed graph of Figure 1 (where edge weights have been indicated in the figure) using Kruskal's algorithm. Make clear in which **order** the edges are picked by the algorithm and draw the minimum weight spanning tree which is finally obtained. Also explicitly write down the instructions of Kruskal's algorithm which prescribe the order in which edges to be included in the minimum spanning tree are picked.

Solution exercise 4: Kruskal's algorithm orders edges from lowest weight to highest weight and then picks edges in that order under the condition that the next edge on the list is only added if there is no cycle formed by adding that edge. Applying this algorithm a possible order for edges to be added is: $\{H, I\}, \{C, F\}, \{D, E\}, \{D, F\}, \{B, E\}, \{E, G\}, \{F, I\}, \{A, C\}$.
The resulting minimum spanning tree is then as follows which has total weight of 40:



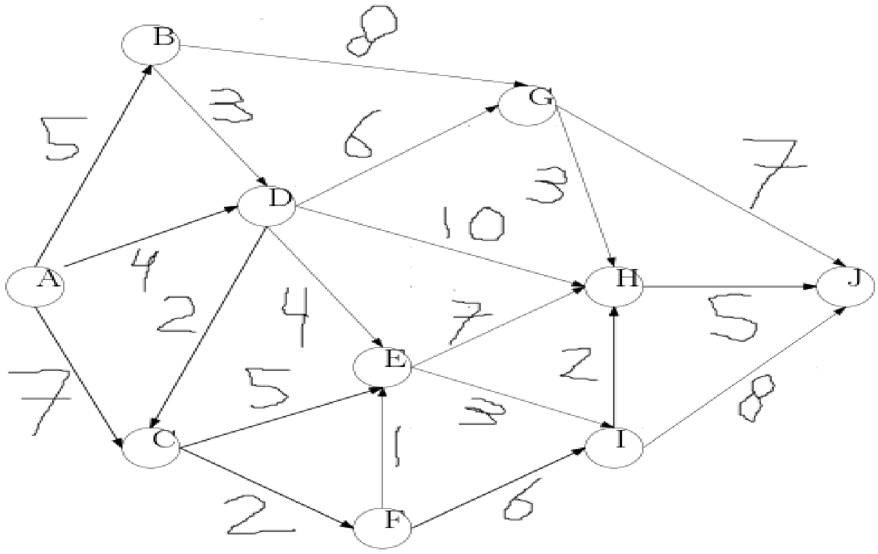


Figure 2: Exercise 5

Exercise 5

Consider the directed acyclic graph as drawn in Figure 2 with lengths of the arcs as indicated in the figure.

(a) **[10 points]** Apply Dijkstra's algorithm to determine the **shortest path** from node A to node J in this acyclic directed graph. It should be clear from the computations you write down that you have applied Dijkstra's algorithm to find the shortest path. After finishing the computations also indicate explicitly the shortest path which is obtained and the length of that shortest path.

(b) **[10 points]** Apply backward recursion to determine the **longest path** from node A to node J in the acyclic directed graph. Make sure it is clear in which order you calculate the function values and explain why the function values are calculated in that order.
Besides the function values calculated by applying backward recursion also indicate explicitly the longest path which is obtained and the length of that longest path.

Solution exercise 5:

(a) The calculations from Dijkstra's algorithm are summarized in the following table where label (x, y) for a node means that x is the shortest distance from A to the node obtained until the current iteration and y is the predecessor of the node on that shortest path to the node. Once the label of a node has become permanent it is no longer written in this table in the following iterations (since it will not change anymore). If the label is followed by * it means that this label becomes permanent in the current iteration.

iteration	B	C	D	E	F	G	H	I	J
1	$(5, A)$	$(7, A)$	$(4, A)^*$						
2	$(5, A)^*$	$(6, D)$		$(8, D)$		$(10, D)$	$(14, D)$		
3		$(6, D)^*$		$(8, D)$		$(10, D)$	$(14, D)$		
4				$(8, D)^*$	$(8, C)$	$(10, D)$	$(14, D)$		
5					$(8, C)^*$	$(10, D)$	$(14, D)$	$(11, E)$	
6						$(10, D)^*$	$(14, D)$	$(11, E)$	
7							$(13, G)$	$(11, E)^*$	$(17, G)$
8							$(13, G)^*$		$(17, G)$
9									$(17, G)^*$

Backtracking from node J we obtain that a shortest path is: $A \to D \to G \to J$ having total length 17.
Remark. In iterations 4 and 5 node E and node F could also be made permanent in other order (thus F before E instead).

(b) In the acyclic directed graph we first number the nodes such that for every arc (i, j) in the graph it holds that $i < j$. Such numbering is for example $A = 1$, $B = 2$, $D = 3$, $C = 4$, $F = 5$, $E = 6$, $I = 7$, $G = 8$, $H = 9$, $J = 10$.

Now define the value function $f(i)$ to be the length of the longest path from node numbered i to destination node $J = 10$. Initialize $f(J) = f(10) = 0$ and compute the other function values by the backward recursion $f(i) = \max_{j:(i,j)\in A}[w(i, j) + f(j)]$.
Then it follows consecutively (doing calculations in reverse order of the numbering of the nodes) that $f(H) = 5$, $f(G) = 8$, $f(I) = 8$, $f(E) = 12$, $f(F) = 14$, $f(C) = 17$, $f(D) = 19$, $f(D) = 17$, $f(B) = 22$, $f(A) = 27$.
Backtracking we obtain as longest path the path $A \to B \to D \to C \to E \to H \to J$. It is easily seen that the length of this path is indeed 27 corresponding with the calculated function value $f(A)$.