Exam Operations Research

Date: April 1, 2022
Time: 15:30 - 17:45

Points per exercise:

- Exercise 1 has a total of 30 points (a(10), b(5), c(5) and d(10)).
- Exercise 2 has a total of 15 points.
- Exercise 3 has a total of 15 points (a(10) and b(5)).
- Exercise 4 has a total of 20 points (a(10) and b(10)).
- Exercise 5 has a total of 10 points.

Thus in total 90 points can be obtained. The exam grade is determined as follows:

$$\text{Exam grade } = 1 + \frac{\text{total number of obtained points}}{10}.$$

- Calculator is allowed.
- This exam consists of 6 pages, including this one.
- The duration of this exam is **2 hours and 15 minutes**.
- Students who have obtained permission for extra time may use an *additional 30 minutes*.

Exercise 1

Consider the following LP which is referred to as the "primal LP".

$$\begin{aligned} \max \quad & z = 3x_1 - 2x_2 + 4x_3 \\ \text{s.t.} \quad & 2x_1 - x_2 + 3x_3 \leq 20 \\ & -x_1 + x_2 - 3x_3 \geq 10 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(a) **[10 points]** Determine the dual of this LP.

(b) **[5 points]** The primal LP has been solved by the simplex method resulting in the following final tableau where s_1 is the slack variable introduced in the first constraint, s_2 is the surplus variable introduced in the second constraint and r_2 is the artificial variable introduced in the second constraint:

Basic	z	x_1	x_2	x_3	s_1	s_2	r_2	value
z	1	0	0	2	1	1	-1	10
x_1	0	1	0	0	1	-1	1	30
x_2	0	0	1	-3	1	-2	2	40

Determine the optimal solution of the primal LP and optimal objective value. Determine also the optimal solution of the dual LP and optimal dual objective value.

(c) **[5 points]** Suppose the right hand sides of both constraints are changed to 30 (instead of respectively 20 and 10). Determine the optimal solution and optimal objective value of this modified LP.

(d) **[10 points]** Now the right hand sides of the constraints are not changed (thus they are respectively 20 and 10 as in the primal LP) but instead the coefficient of variable x_1 in the objective function is changed from 3 to 1. Adjust the simplex tableau accordingly and apply the simplex method to determine the optimal solution and optimal objective value of this modified LP. It will require one pivot step to find the new optimal solution. Also explain why it follows from the simplex tableau which you obtain after one pivot step that the corresponding solution is the optimal solution of the modified LP.

Exercise 2

In some national soccer competition there are seven postponed matches and there is an upcoming weekend designated to play postponed matches. The objective of the competition manager is that as many as possible of these postponed matches will be played during that weekend. However for several reasons there are restrictions on these matches to be played in that weekend. The restrictions for that weekend are as follows:

- If matches 1 and 4 are both played then match 2 can not be played.
- If match 6 is not played then match 5 has to be played.
- At least two of the three matches 3, 4 and 7 have to be played.
- If match 3 is played then matches 1 and 6 can not be played.
- If match 7 is played then at most one of the three matches 2, 4 and 5 can be played.

(a) **[15 points]** Formulate the problem of maximizing the number of postponed matches to be played during that weekend under the given restrictions as an integer linear program (ILP). Explain all variables and constraints in your ILP formulation of this problem.

Exercise 3

Consider the following ILP (Integer Linear Program) with three nonnegative integer decision variables x_1, x_2, x_3.

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 5x_3 \\ \text{s.t.} \quad & 4x_1 + 3x_2 + 2x_3 \leq 13 \\ & 2x_1 + 3x_2 + 4x_3 \leq 11 \\ & x_1, x_2, x_3 \geq 0 \\ & x_1, x_2, x_3 \text{ integer} \end{aligned}$$

The LP relaxation of the ILP has been solved by the simplex method resulting in the following final simplex tableau.

Basic	z	x_1	x_2	x_3	s_1	s_2	value
z	1	0	1	0	$\frac{5}{6}$	$\frac{5}{6}$	20
x_1	0	1	0.50	0	$\frac{1}{3}$	$-\frac{1}{6}$	2.50
x_3	0	0	0.50	1	$-\frac{1}{6}$	$\frac{1}{3}$	1.50

The solution obtained from this simplex tableau is not integer and thus not feasible for the original ILP. Branching on decision variable x_1 results in two subproblems. For subproblem 1 the constraint $x_1 \leq 2$ is added to the problem and for subproblem 2 the constraint $x_1 \geq 3$ is added to the problem.

(a) **[10 points]** First write down the LP relaxation of subproblem 1 in standard form. Next solve this problem by applying the dual simplex method.
Instruction: To reduce computation time make use of the above given final simplex tableau for the LP relaxation of the original ILP.

(b) **[5 points]** The LP relaxation of subproblem 2 has also been solved resulting in a solution where the decision variables x_i have values $x_1 = 3$, $x_2 = 0$ and $x_3 = 0.50$. Determine an optimal solution of the original ILP by the branch-and-bound method. Explain why that solution is optimal for the original ILP.
If in (a) you were not able to solve the LP relaxation of subproblem 1 you may assume that $x_1 = 1, x_2 = 3, x_3 = 0$ is an optimal solution of that problem to continue the branch-and-bound method.

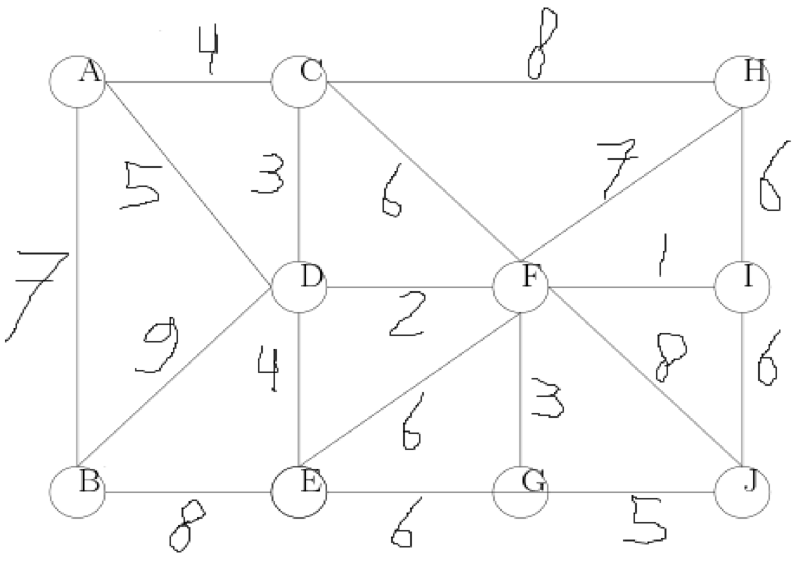

Figure 1: non-directed graph exercise 4a

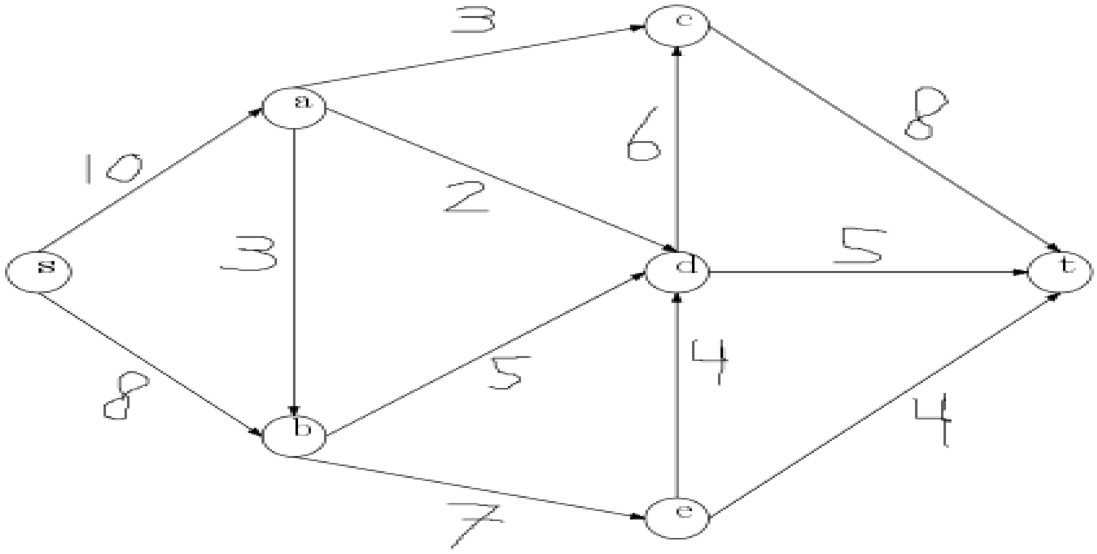

Figure 2: directed graph exercise 4b

Exercise 4

(a) **[10 points]** Consider the problem of finding a minimum weight spanning tree in the non-directed graph of Figure 1 (where edge weights have been indicated) using Prim's algorithm starting from the tree containing only node A. Make clear in which **order** the edges are picked by the algorithm and draw the minimum weight spanning tree which is finally obtained. Explain the order in which edges to be included in the minimum spanning tree are picked by this algorithm.

(b) **[10 points]** Consider the maximum flow problem shown in the directed graph of Figure 2 where the arc capacities are indicated by the numbers near the arcs.
Let the current flow f (which is feasible but not maximal) be as follows:
$f_{sa} = 6,\ f_{sb} = 8,\ f_{ab} = 1,\ f_{ac} = 3,\ f_{ad} = 2,\ f_{bd} = 5,\ f_{be} = 4,\ f_{ct} = 5,\ f_{dc} = 2,\ f_{dt} = 5,\ f_{ed} = 0,\ f_{et} = 4$
and no flow on other arcs. Draw the residual graph D^f corresponding to this flow f. Continue from this residual graph the Ford-Fulkerson algorithm to determine a maximum flow from source node s to sink node t. State the value of the flow and show that it is maximal by providing an s-t cut of the same value.

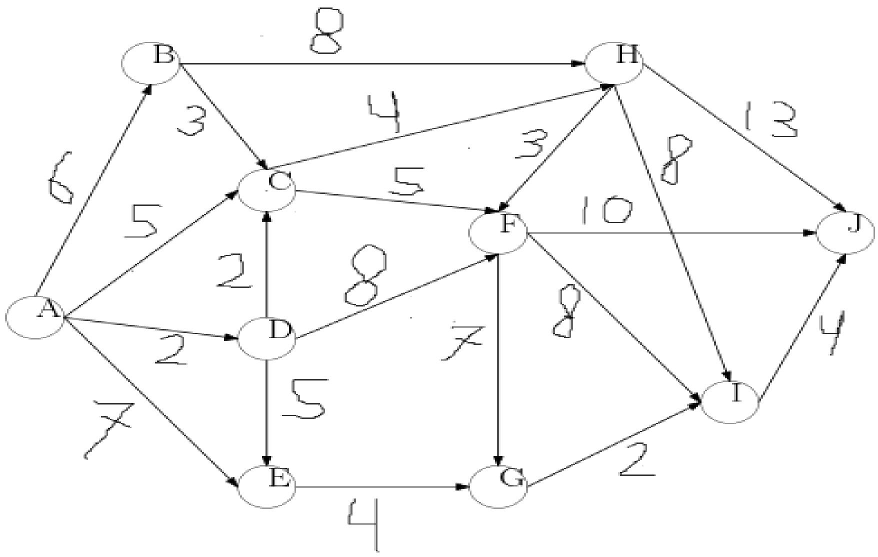

Figure 3: acyclic directed graph exercise 5a

Exercise 5

Consider the acyclic directed graph of Figure 3 with lengths of the arcs as indicated in the graph.

(a) **[10 points]**. Apply dynamic programming to determine the **longest path** from node A to node J in this directed graph. Define an appropriate value function and use backward recursion to compute for all states the function value. Determine the length of the longest path and make clear which path is the longest path you have obtained.