Exam Operations Research

Date: April 1, 2022
Time: 15:30 - 17:45

Points per exercise:

- Exercise 1 has a total of 30 points (a(10), b(5), c(5) and d(10)).
- Exercise 2 has a total of 15 points.
- Exercise 3 has a total of 15 points (a(10) and b(5)).
- Exercise 4 has a total of 20 points (a(10) and b(10)).
- Exercise 5 has a total of 10 points.

Thus in total 90 points can be obtained. The exam grade is determined as follows:

$$\text{Exam grade } = 1 + \frac{\text{total number of obtained points}}{10}.$$

- Calculator is allowed.
- This exam consists of 6 pages, including this one.
- The duration of this exam is **2 hours and 15 minutes**.
- Students who have obtained permission for extra time may use an *additional 30 minutes*.

Exercise 1

Consider the following LP which is referred to as the "primal LP".

$$\begin{aligned} \max \quad & z = 3x_1 - 2x_2 + 4x_3 \\ \text{s.t.} \quad & 2x_1 - x_2 + 3x_3 \leq 20 \\ & -x_1 + x_2 - 3x_3 \geq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(a) **[10 points]** Determine the dual of this LP.

(b) **[5 points]** The primal LP has been solved by the simplex method resulting in the following final tableau where s_1 is the slack variable introduced in the first constraint, s_2 is the surplus variable introduced in the second constraint and r_2 is the artificial variable introduced in the second constraint:

Basic	z	x_1	x_2	x_3	s_1	s_2	r_2	value
z	1	0	0	2	1	1	-1	10
x_1	0	1	0	0	1	-1	1	30
x_2	0	0	1	-3	1	-2	2	40

Determine the optimal solution of the primal LP and optimal objective value. Determine also the optimal solution of the dual LP and optimal dual objective value.

(c) **[5 points]** Suppose the right hand sides of both constraints are changed to 30 (instead of respectively 20 and 10). Determine the optimal solution and optimal objective value of this modified LP.

(d) **[10 points]** Now the right hand sides of the constraints are not changed (thus they are respectively 20 and 10 as in the primal LP) but instead the coefficient of variable x_1 in the objective function is changed from 3 to 1. Adjust the simplex tableau accordingly and apply the simplex method to determine the optimal solution and optimal objective value of this modified LP. It will require one pivot step to find the new optimal solution. Also explain why it follows from the simplex tableau which you obtain after one pivot step that the corresponding solution is the optimal solution of the modified LP.

Solution Exercise 1:

(a) The dual LP is:

$$\begin{aligned} \min \quad & w = 20y_1 + 10y_2 \\ \text{s.t.} \quad & 2y_1 - y_2 \geq 3 \\ & -y_1 + y_2 \geq -2 \\ & 3y_1 - 3y_2 \geq 4 \\ & y_1 \geq 0, y_2 \leq 0 \end{aligned}$$

(b) The optimal solution of the primal LP is $x_1^* = 30$, $x_2^* = 40$, $x_3^* = 0$ and optimal objective value $z^* = 10$. The optimal solution of the dual LP is $y_1^* = 1$, $y_2^* = -1$ with optimal dual objective value $w^* = 10$.

(c) The change in the right hand side of the first constraint is $\Delta_1 = 30 - 20 = 10$ and in the second constraint is $\Delta_2 = 30 - 10 = 20$. Thus as corresponding basic solution of the modified LP we obtain $x_1^* = 30 + \Delta_1 + \Delta_2 = 60$ and $x_2^* = 40 + \Delta_1 + 2\Delta_2 = 90$. For the nonbasic variable x_3 we still have $x_3^* = 0$. Notice that this solution is feasible and thus it is optimal for the modified LP. It follows that the new optimal objective value is $z^* = 10 + \Delta_1 - \Delta_2 = 0$.

(d) Since the coefficient of variable x_1 in the objective function is decreased by 2 it follows that in the simplex tableau the coefficient of x_1 in the z -row increases by 2. This gives the following simplex tableau:

Basic	z	x_1	x_2	x_3	s_1	s_2	r_2	value
z	1	2	0	2	1	1	−1	10
x_1	0	1	0	0	1	−1	1	30
x_2	0	0	1	−3	1	−2	2	40

Since x_1 is basic variable the coefficient of x_1 in the z-row should become 0 which is possible by subtracting two times the x_1-row from the z-row resulting in the following tableau:

Basic	z	x_1	x_2	x_3	s_1	s_2	r_2	value
z	1	0	0	2	−1	3	−3	−50
x_1	0	1	0	0	1	−1	1	30
x_2	0	0	1	−3	1	−2	2	40

Since the coefficient of s_1 in the z-row is now negative this solution is not optimal and s_1 should become basic variable. Then x_1 has to leave the basis by the ratio test since $\frac{30}{1} < \frac{40}{1}$. Performing the corresponding pivot step gives the following simplex tableau:

Basic	z	x_1	x_2	x_3	s_1	s_2	r_2	value
z	1	1	0	2	0	2	−2	−20
s_1	0	1	0	0	1	−1	1	30
x_2	0	−1	1	−3	0	−1	1	10

The corresponding solution is optimal since all coefficients (except for the artificial variable r_2) in the z-row are now nonnegative. Thus an optimal solution of the modified LP is: $x_1^* = 0$, $x_2^* = 10$, $x_3^* = 0$ with optimal objective value $z^* = -20$.

Exercise 2

In some national soccer competition there are seven postponed matches and there is an upcoming weekend designated to play postponed matches. The objective of the competition manager is that as many as possible of these postponed matches will be played during that weekend. However for several reasons there are restrictions on these matches to be played in that weekend. The restrictions for that weekend are as follows:

- If matches 1 and 4 are both played then match 2 can not be played.
- If match 6 is not played then match 5 has to be played.
- At least two of the three matches 3, 4 and 7 have to be played.
- If match 3 is played then matches 1 and 6 can not be played.
- If match 7 is played then at most one of the three matches 2, 4 and 5 can be played.

(a) **[15 points]** Formulate the problem of maximizing the number of postponed matches to be played during that weekend under the given restrictions as an integer linear program (ILP). Explain all variables and constraints in your ILP formulation of this problem.

Solution Exercise 2:
Define binary variables x_i for $i = 1, 2, \dots, 7$ where $x_i = 1$ if match i is scheduled to be played in that weekend and $x_i = 0$ otherwise. Using these decision variables the problem can be formulated as the following ILP:

$$\max \quad z = \sum_{i=1}^{7} x_i$$

s.t.	$x_1 + x_2 + x_4 \leq 2$	restriction 1
	$x_5 + x_6 \geq 1$	restriction 2
	$x_3 + x_4 + x_7 \geq 2$	restriction 3
	$x_1 + x_3 \leq 1$	part restriction 4
	$x_3 + x_6 \leq 1$	part restriction 4
	$x_2 + x_4 + x_5 \leq 1 + 2(1 - x_7)$	restriction 5
	All variables $x_i \in \{0, 1\}$	

Exercise 3

Consider the following ILP (Integer Linear Program) with three nonnegative integer decision variables x_1, x_2, x_3.

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 5x_3 \\ \text{s.t.} \quad & 4x_1 + 3x_2 + 2x_3 \leq 13 \\ & 2x_1 + 3x_2 + 4x_3 \leq 11 \\ & x_1, x_2, x_3 \geq 0 \\ & x_1, x_2, x_3 \text{ integer} \end{aligned}$$

The LP relaxation of the ILP has been solved by the simplex method resulting in the following final simplex tableau.

Basic	z	x_1	x_2	x_3	s_1	s_2	value
z	1	0	1	0	$\frac{5}{6}$	$\frac{5}{6}$	20
x_1	0	1	0.50	0	$\frac{1}{3}$	$-\frac{1}{6}$	2.50
x_3	0	0	0.50	1	$-\frac{1}{6}$	$\frac{1}{3}$	1.50

The solution obtained from this simplex tableau is not integer and thus not feasible for the original ILP. Branching on decision variable x_1 results in two subproblems. For subproblem 1 the constraint $x_1 \leq 2$ is added to the problem and for subproblem 2 the constraint $x_1 \geq 3$ is added to the problem.

(a) **[10 points]** First write down the LP relaxation of subproblem 1 in standard form. Next solve this problem by applying the dual simplex method.
Instruction: To reduce computation time make use of the above given final simplex tableau for the LP relaxation of the original ILP.

(b) **[5 points]** The LP relaxation of subproblem 2 has also been solved resulting in a solution where the decision variables x_i have values $x_1 = 3$, $x_2 = 0$ and $x_3 = 0.50$. Determine an optimal solution of the original ILP by the branch-and-bound method. Explain why that solution is optimal for the original ILP.
If in (a) you were not able to solve the LP relaxation of subproblem 1 you may assume that $x_1 = 1, x_2 = 3, x_3 = 0$ is an optimal solution of that problem to continue the branch-and-bound method.

Solution exercise 3:

(a) The LP relaxation of subproblem 1 in standard form is as follows:

$$\begin{array}{rl} \max & z = 5x_1 + 4x_2 + 5x_3 \\ \text{s.t.} & 4x_1 + 3x_2 + 2x_3 + s_1 \qquad = 13 \\ & 2x_1 + 3x_2 + 4x_3 + \quad +s_2 \quad = 11 \\ & x_1 \qquad\qquad\qquad + s_3 = 2 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

In this standard form s_1, s_2 and s_3 are slack variables. The extra equation $x_1 + s_3 = 2$ is added to the final simplex tableau given in the exercise and a column for the new slack variable s_3 is added. This give the following tableau:

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	value
z	1	0	1	0	$\frac{5}{6}$	$\frac{5}{6}$	0	20
x_1	0	1	0.50	0	$\frac{1}{3}$	$-\frac{1}{6}$	0	2.50
x_3	0	0	0.50	1	$-\frac{1}{6}$	$\frac{1}{3}$	0	1.50
s_3	0	1	0	0	0	0	1	2

To get the column of basic variable x_1 correct the x_1-row is subtracted from the s_3-row resulting in the following tableau:

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	value
z	1	0	1	0	$\frac{5}{6}$	$\frac{5}{6}$	0	20
x_1	0	1	0.50	0	$\frac{1}{3}$	$-\frac{1}{6}$	0	2.50
x_3	0	0	0.50	1	$-\frac{1}{6}$	$\frac{1}{3}$	0	1.50
s_3	0	0	-0.50	0	$-\frac{1}{3}$	$\frac{1}{6}$	1	-0.50

Since the value of s_3 is now negative s_3 should leave the basis and according to the ratio test for the dual simplex method variable x_2 then enters the basis. Performing the corresponding pivot step results in the following tableau:

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	value
z	1	0	0	0	$\frac{1}{6}$	$\frac{7}{6}$	2	19
x_1	0	1	0	0	0	0	1	2
x_3	0	0	0	1	-0.50	0.50	1	1
x_2	0	0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	-2	1

The corresponding solution $x_1 = 2$, $x_2 = 1$, $x_3 = 1$ is optimal with value $z = 19$.

(b) The solution $x_1 = 2$, $x_2 = 1$, $x_3 = 1$ which was obtained in (a) is integer and thus feasible for the original ILP. Thus the corresponding objective value $z = 19$ is a lower bound for the optimal objectuive value of the original ILP. The given optimal solution $x_1 = 3$, $x_2 = 0$ and $x_3 = 0.50$ of the LP relaxation of subproblem 2 has value $z = 5 \times 3 + 4 \times 0 + 5 \times 0.50 = 17.5$. Hence subproblem 2 can not have a solution with value better than 17.5 (and thus also not better than 17 since the value of a feasible integer solution is integer). Hence the optimal solution $x_1 = 2$, $x_2 = 1$, $x_3 = 1$ for subproblem 1 with objective value $z = 19$ is optimal for the original ILP since subproblem 2 can not have a better solution.

Remark: If the optimal solution for (a) was not obtained and therefore instead the solution $x_1 = 1, x_2 = 3, x_3 = 0$ for subproblem 1 is assumed to be optimal it can similarly be concluded that that solution would be optimal for the original ILP since the objective value for that solution is $z = 17$ and then it is still not possible that subproblem 2 contains a better solution.

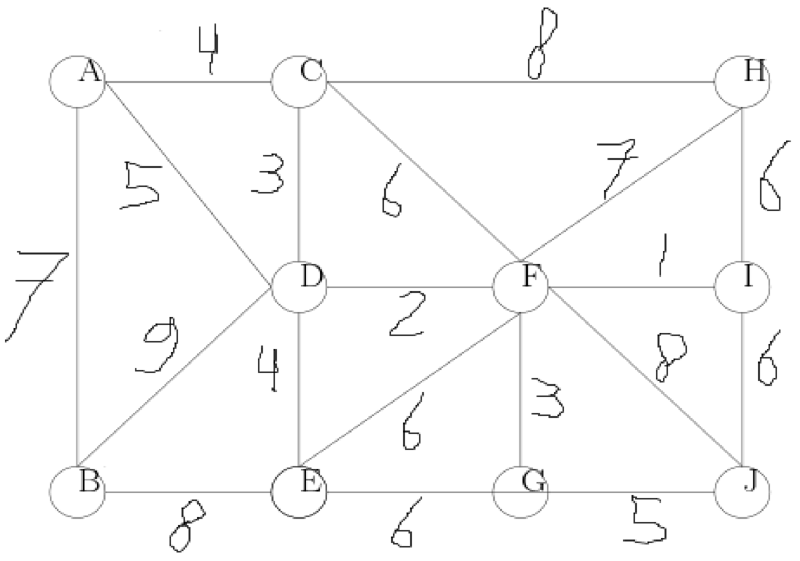


Figure 1: non-directed graph exercise 4a

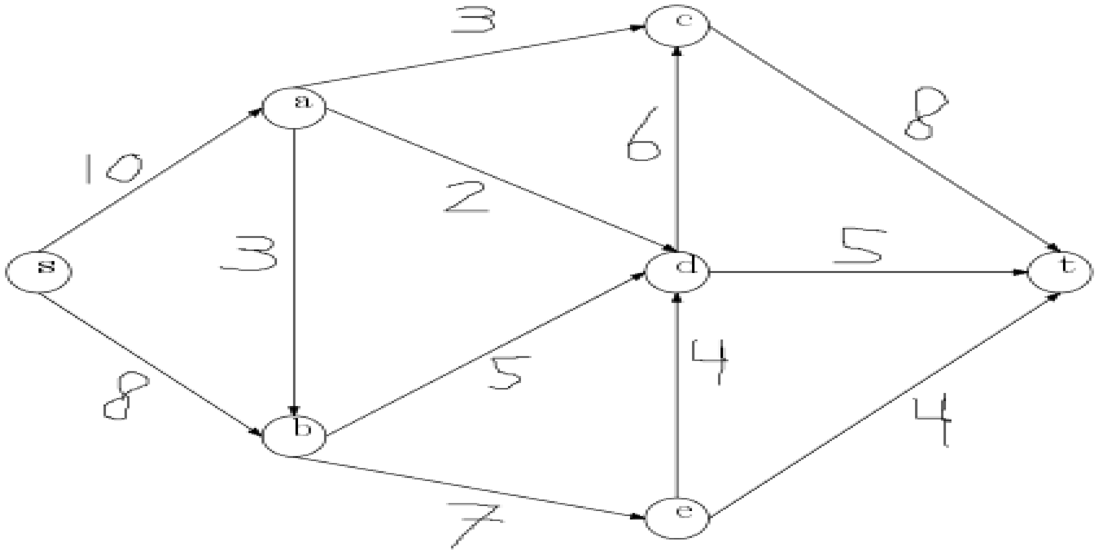


Figure 2: directed graph exercise 4b

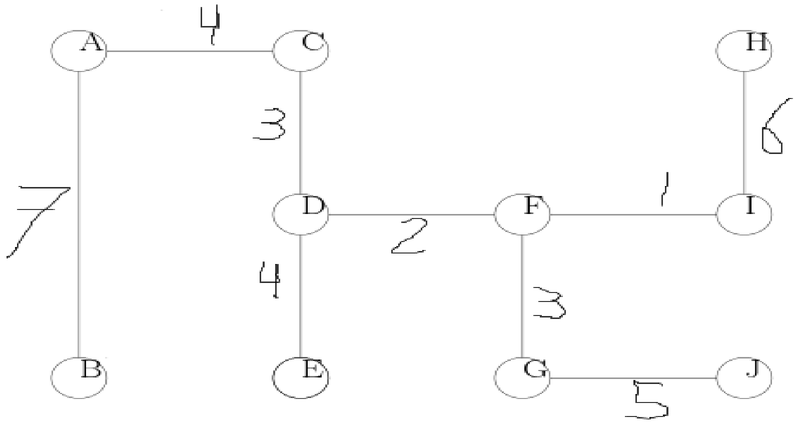
Exercise 4

(a) **[10 points]** Consider the problem of finding a minimum weight spanning tree in the non-directed graph of Figure 1 (where edge weights have been indicated) using Prim's algorithm starting from the tree containing only node A. Make clear in which **order** the edges are picked by the algorithm and draw the minimum weight spanning tree which is finally obtained. Explain the order in which edges to be included in the minimum spanning tree are picked by this algorithm.

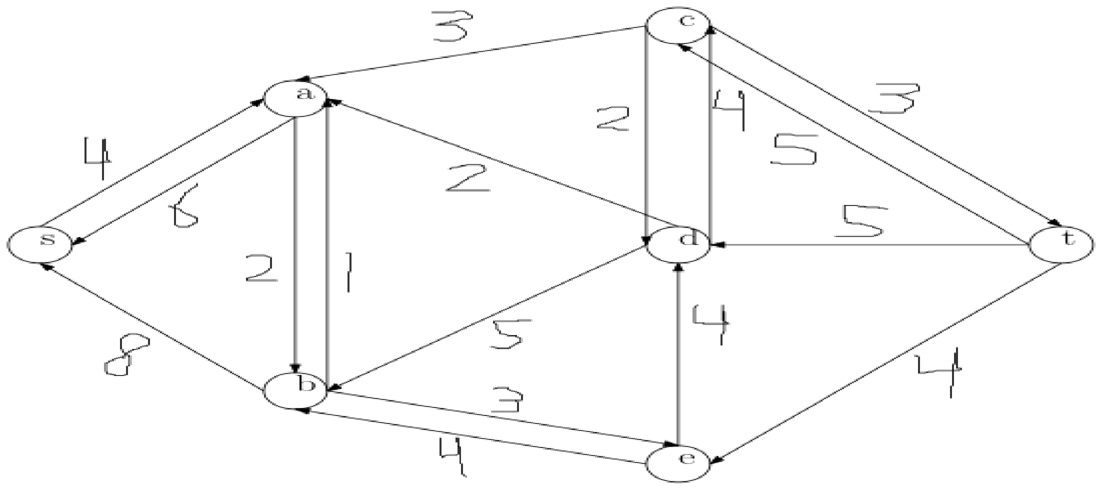
(b) **[10 points]** Consider the maximum flow problem shown in the directed graph of Figure 2 where the arc capacities are indicated by the numbers near the arcs.
Let the current flow f (which is feasible but not maximal) be as follows:
$f_{sa} = 6,\ f_{sb} = 8,\ f_{ab} = 1,\ f_{ac} = 3,\ f_{ad} = 2,\ f_{bd} = 5,\ f_{be} = 4,\ f_{ct} = 5,\ f_{dc} = 2,\ f_{dt} = 5,\ f_{ed} = 0,\ f_{et} = 4$
and no flow on other arcs. Draw the residual graph D^f corresponding to this flow f. Continue from this residual graph the Ford-Fulkerson algorithm to determine a maximum flow from source node s to sink node t. State the value of the flow and show that it is maximal by providing an s-t cut of the same value.

Solution exercise 4:

(a) Prim's algorithm adds an edge of lowest weight under the condition that the edge should be connected to the existing tree and there is no cycle formed by adding that edge. Applying this algorithm the following edges are added in the following order: $\{A,C\}, \{C,D\}, \{D,F\}, \{F,I\}, \{F,G\}, \{D,E\}, \{G,J\}, \{H,I\}, \{A,B\}$. The resulting minimum spanning tree is then as follows:



(b) The residual graph for the given flow is as follows:



An augmenting path in the above residual graph is: $s \to a \to b \to e \to d \to c \to t$ on which 2 units of extra flow can be pushed. Pushing this extra flow of 2 units will remove the arc from a to b in the next residual graph. It is easily seen that in that next residual graph there will be no longer a path from s to t because there are no arcs which leave the subset $\{s,a\}$ of nodes. Thus the resulting flow after pushing 2 extra units on the augmenting path should be maximal. This resulting flow is:
$f_{sa} = 8,\ f_{sb} = 8,\ f_{ab} = 3,\ f_{ac} = 3,\ f_{ad} = 2,\ f_{bd} = 5,\ f_{be} = 6,\ f_{ct} = 7,\ f_{dc} = 4,\ f_{dt} = 5,\ f_{ed} = 2,\ f_{et} = 4$ and no flow on other arcs having value 16.

To show that this is indeed a maximal flow an $s-t$ cut in the original graph of the same total capacity of 16 should be provided. This is the cut consisting of the arcs $\{(s,b),(a,b),(a,c),(a,d)\}$ which indeed has a total capacity of $8+3+3+2 = 16$. This minimum cut is the cut between the nodes $\{s,a\}$ and the nodes $\{b,c,d,e,t\}$.

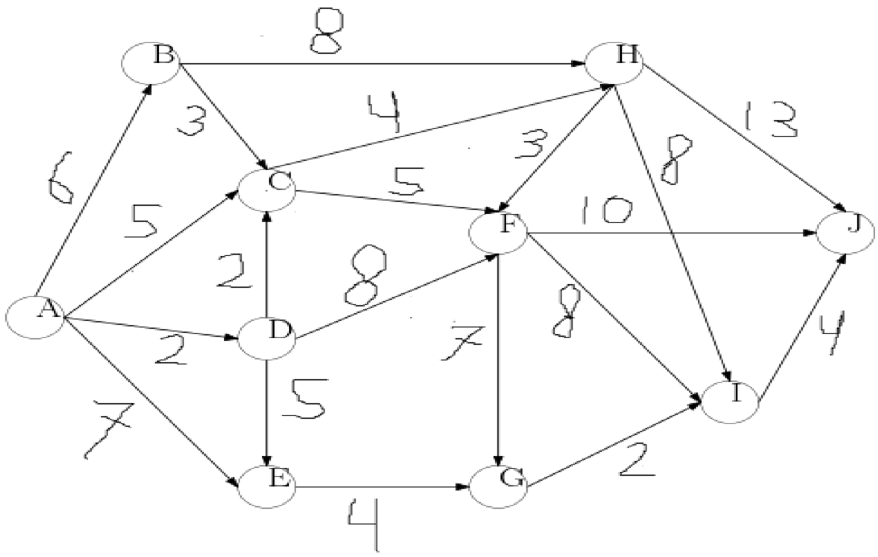


Figure 3: acyclic directed graph exercise 5a

Exercise 5

Consider the acyclic directed graph of Figure 3 with lengths of the arcs as indicated in the graph.

(a) **[10 points]**. Apply dynamic programming to determine the **longest path** from node A to node J in this directed graph. Define an appropriate value function and use backward recursion to compute for all states the function value. Determine the length of the longest path and make clear which path is the longest path you have obtained.

Solution exercise 5:
Because the directed graph is acyclic we can number the nodes in the graph such that there are only forward arcs with respect to that numbering. After such numbering of the nodes backward recursion can be applied using the numbering of the nodes. Such a numbering which is applicable for backward recursion is $A = 1$, $B = 2$, $D = 3$, $E = 4$, $C = 5$, $H = 6$, $F = 7$, $G = 8$, $I = 9$, $J = 10$.
Define the value function $f(i)$ to be the length of the longest path from node numbered i to destination node $J = 10$. Initialize $f(J) = f(10) = 0$ and compute the other function values in backward order by the recursion $f(i) = \max_{j:(i,j)\in A}[w(i,j)+f(j)]$. Then it follows consecutively (doing calculations in reverse order of the numbering of the nodes) that $f(I) = 4$, $f(G) = 6$, $f(F) = 13$, $f(H) = 16$, $f(C) = 20$, $f(E) = 10$, $f(D) = 22$, $f(B) = 24$, $f(A) = 30$.
Backtracking we obtain as longest path the path $A \to B \to H \to F \to G \to I \to J$. It is easily checked that the length of this path is indeed 30 corresponding with the function value $f(A)$.