Resit Exam Operations Research

Date: June 2, 2021
Time: 12:15 - 14:30

Points per exercise:

- Exercise 1 has a total of 30 points (a(10), b(5), c(5) and d(10)).
- Exercise 2 has a total of 15 points.
- Exercise 3 has a total of 15 points (a(5) and b(10)).
- Exercise 4 has a total of 10 points.
- Exercise 5 has a total of 20 points (a(10) and b(10)).

Thus in total 90 points can be obtained. The exam grade is determined as follows:

$$\text{Exam grade } = 1 + \frac{\text{total number of obtained points}}{10}.$$

- Calculator is allowed.
- This exam consists of 5 pages, including this one.
- The duration of this exam is **2 hours and 15 minutes**. When you finish working you notify the host of your meeting. Within 15 minutes after you finish working you have to upload the scanned pdf file in Canvas in the assignment for the exam. After uploading the pdf file you notify the host again.
- Students who have obtained permission for extra time may use an *additional 30 minutes*.

Exercise 1

Consider the following LP which is referred to as the "primal LP".

$$
\begin{aligned}
\min \quad & w = 2x_1 + 3x_2 - x_3 + x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + 3x_4 \geq 16 \\
& 2x_1 + x_2 + x_3 + 2x_4 = 14 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

(a) **[10 points]** Determine the dual of this LP.

(b) **[5 points]** The primal LP has been solved by the simplex method resulting in the following final tableau where s_1 is the surplus variable introduced in the first constraint, r_1 is the artificial variable introduced in the first constraint and r_2 is the artificial variable introduced in the second constraint.

Basic	w	x_1	x_2	x_3	x_4	s_1	r_1	r_2	value
w	1	-2.20	-2.20	0	0	-0.60	0.60	-0.40	4
x_4	0	0.60	0.60	0	1	-0.20	0.20	0.20	6
x_3	0	0.80	-0.20	1	0	0.40	-0.40	0.60	2

Determine the optimal solution of the primal LP and optimal objective value. Determine also the optimal solution of the dual LP and optimal dual objective function.

(c) **[5 points]** Suppose the right hand sides of both constraints are changed to 20 (instead of respectively 16 and 14). Determine the optimal solution and optimal objective value of this modified LP.

(d) **[10 points]** Now assume the right hand side of the first constraint is changed to 26 (instead of 16) and the right hand side of the second constraint is 14 (thus the same as in the original primal LP). Apply the dual simplex method to determine the optimal solution and optimal objective value of this modified LP.
Instruction: Since artificial variables should never become basic variables when applying the dual simplex method the columns of the two artificial variables can be omitted from the simplex tableau when you have chosen the pivot row. Moreover, it will require only one pivot step to find the new optimal solution.

Solution exercise 1:

(a) The dual LP is:

$$\begin{aligned} \max \quad & z = 16y_1 + 14y_2 \\ \text{s.t.} \quad & y_1 + 2y_2 \leq 2 \\ & 2y_1 + y_2 \leq 3 \\ & -y_1 + y_2 \leq -1 \\ & 3y_1 + 2y_2 \leq 1 \\ & y_1 \geq 0, y_2 \text{ unrestricted} \end{aligned}$$

(b) The optimal solution of the primal LP is $x_1^* = 0$, $x_2^* = 0$, $x_3^* = 2$, $x_4^* = 6$ and optimal objective value $w^* = 4$. The optimal solution of the dual LP is $y_1^* = 0.60$, $y_2^* = -0.40$ with optimal dual objective value $z^* = 4$.

(c) The change in the right hand side of the first constraint is $\Delta_1 = 20-16 = 4$ and in the second constraint the change is $\Delta_2 = 20 - 14 = 6$. Thus as corresponding basic solution of the modified LP we obtain $x_4^* = 6 + 0.20\Delta_1 + 0.20\Delta_2 = 8$ and $x_3^* = 2 - 0.40\Delta_1 + 0.60\Delta_2 = 4$ and for the nonbasic variables we still have $x_1^* = x_2^* = 0$. Notice that this solution is feasible and thus optimal for the modified LP. It follows that the new optimal objective value is $w^* = 4 + 0.60\Delta_1 + -0.40\Delta_2 = 4$.

(d) Now the change in right hand side of the first constraint is $\Delta_1 = 26 - 16 = 10$ and in the second constraint the change is $\Delta_2 = 14 - 14 = 0$. Computing the corresponding new solution similar as in 1c we obtain $x_4 = 6+0.20\Delta_1+0.20\Delta_2 = 8$ and $x_3 = 2-0.40\Delta_1+0.60\Delta_2 = -2$. For the nonbasic variables we still have $x_1 = x_2 = 0$ and the new objective value is $w = 4+0.60\Delta_1+-0.40\Delta_2 = 10$. However, this solution is not feasible (and thus also not optimal) since x_3 is negative and therefore the dual simplex method has to be applied.

Omitting the columns of the two artificial variables and filling in the values calculated above in the value column gives the following simplex tableau:

Basic	w	x_1	x_2	x_3	x_4	s_1	value
w	1	-2.20	-2.20	0	0	-0.60	10
x_4	0	0.60	0.60	0	1	-0.20	8
x_3	0	0.80	-0.20	1	0	0.40	-2

Since x_3 is negative and has to leave the basis it follows that x_2 will enter the basis since it is the only variable with negative coefficient in the x_3-row. We divide the row of new basis variable x_2 by -0.20 to get a 1 at the pivot position. Then we have the following simplex tableau:

Basic	w	x_1	x_2	x_3	x_4	s_1	value
w	1	-2.20	-2.20	0	0	-0.60	10
x_4	0	0.60	0.60	0	1	-0.20	8
x_2	0	-4	1	-5	0	-2	10

Now the column of new basis variable x_2 has still to be transformed to unit vector. Therefore 2.2 times x_2-row is added to the w-row and 0.6 times x_2-row is subtracted from the x_4-row. We obtain the following tableau:

Basic	w	x_1	x_2	x_3	x_4	s_1	value
w	1	-11	0	-11	0	-5	32
x_4	0	3	0	3	1	1	2
x_2	0	-4	1	-5	0	-2	10

From this tableau it can be concluded that the optimal solution of the modified LP is $x_1^* = 0$, $x_2^* = 10$, $x_3^* = 0$, $x_4^* = 2$ with optimal objective value $w^* = 32$.

Exercise 2

Bill has to buy 5 items. He can go to 4 shops which sell these items. However not every item is available in every shop. In the table you see for each shop j ($j = 1, 2, 3, 4$) which of the items i ($i = 1, 2, 3, 4, 5$) can be bought in that shop.

Shop	Items available
1	$1, 2, 3$
2	$2, 3, 4, 5$
3	$1, 3, 4, 5$
4	$1, 2, 3, 5$

In shop j the price for item i (if item i is available in shop j) is $p_{ij} > 0$. The travelling expenses for Bill to visit shop j (to buy one or more items at that shop) is $c_j > 0$. Bill wants to obtain all 5 items minimizing the total amount of prices paid and travelling expenses made.

(a) **[15 points]** Formulate Bill's problem of minimizing the total costs to obtain all 5 items as an integer linear program (ILP). Explain all variables and constraints in your ILP formulation of this problem.

Solution exercise 2: Define binary variables x_{ij} for $i = 1, 2, \ldots, 5$ and $j = 1, 2, 3, 4$ where $x_{ij} = 1$ if Bill buys item i in shop j. Also define binary variables y_j for $j = 1, 2, 3, 4$ where $y_j = 1$ if Bill buys at least one item in shop j (and thus he will then travel to shop j). Using these decision variables the problem can be formulated as the following ILP:

$$\min \quad w = \sum_{i=1}^{5}\sum_{j=1}^{4} p_{ij}x_{ij} + \sum_{j=1}^{4} c_j y_j$$

$$\text{s.t.} \quad \sum_{j=1}^{4} x_{ij} = 1 \text{ for } i = 1, 2, \ldots, 5$$

$$My_j \geq \sum_{i=1}^{5} x_{ij} \text{ for } j = 1, 2, 3, 4$$

$$x_{41} = 0,\ x_{51} = 0, x_{12} = 0, x_{23} = 0, x_{44} = 0$$

$$\text{All variables } x_{ij} \in \{0, 1\}, y_j \in \{0, 1\}$$

The first constraints make sure that every item i is bought exactly once by Bill. The second constraints make sure that Bill travels to shop j if he buys at least one item in shop j. Number M should be chosen large enough such that the constraint is always satisfied if $y_j = 1$. In this case $M = 5$ is large enough since the right hand side can never be larger than 5. The third constraints make sure that Bill can not buy an item in a shop where that item is not available. The last constraints are because all x_{ij} and y_j variables are binary variables.

Exercise 3

Consider the following ILP with four binary decision variables x_1, x_2, x_3, x_4.

$$\begin{aligned} \max \quad & z = 13x_1 + 10x_2 + 19x_3 + 25x_4 \\ \text{s.t.} \quad & 6x_1 + 5x_2 + 9x_3 + 12x_4 \leq 18 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

(a) **[5 points]** Write down the LP relaxation of this ILP and determine the unique optimal solution of the LP relaxation. Also determine the corresponding optimal objective value of the LP relaxation.
Instruction: The special form of the ILP makes it possible to quickly determine the optimal solution of the LP relaxation. It is not needed to apply the simplex method.

(b) **[10 points]** Solve the ILP by applying the **branch-and-bound** method. Clearly indicate in which order you compute the nodes of the search tree and where you prune the search tree. Also indicate which pruning criterion you use when you prune.

Solution exercise 3:

(a) The ILP has the following LP relaxation:

$$\begin{aligned} \max \quad & z = 13x_1 + 10x_2 + 19x_3 + 25x_4 \\ \text{s.t.} \quad & 6x_1 + 5x_2 + 9x_3 + 12x_4 \leq 18 \\ & 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3, 4 \end{aligned}$$

Notice that the problem is an $0-1$ knapsack problem and thus the optimal solution of the LP relaxation can be found by ordering the four variables according to value-weight ratio and then applying the greedy algorithm. Since $\frac{13}{6} > \frac{19}{9} > \frac{25}{12} > \frac{10}{5}$ the greedy algorithm gives as optimal solution of the LP relaxation $x_1 = 1$, $x_3 = 1$, $x_4 = \frac{3}{12} = \frac{1}{4}$, $x_2 = 0$ with corresponding optimal objective value $z^* = 38\frac{1}{4}$.

(b) From (a) we already have for starting node 0 the upperbound $z^0 = 38\frac{1}{4}$ and from starting node 0 we have to branch on the fractional variable x_4.
Put $x_4 = 0$ for subproblem 1 (corresponding to node 1 in the search tree). Then the LP relaxation has optimal solution $x_4 = 0, x_1 = 1, x_3 = 1, x_2 = \frac{3}{5}$ with value $z^1 = 38$ which is an upper bound for any feasible solutions obtained from node 1. Put $x_4 = 1$ for subproblem 2 (corresponding to node 2 in the search tree). The LP relaxation in node 2 has optimal solution $x_4 = 1$, $x_1 = 1$, $x_3 = 0$, $x_2 = 0$ with value $z^2 = 38$. This solution is feasible for the original ILP and thus the objective value 38 is a lowerbound for the original ILP. Moreover, for node 2 an optimal solution has been obtained and we do not have to branch further from node 2. Returning to node 1 we already had the upper bound 38 and thus from node 1 we can never obtain a better solution than the feasible solution already obtained in node 2. Hence we can also prune in node 1 and conclude that $x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1$ is an optimal solution of the ILP and the optimal objective value of the ILP is $z^* = 38$.

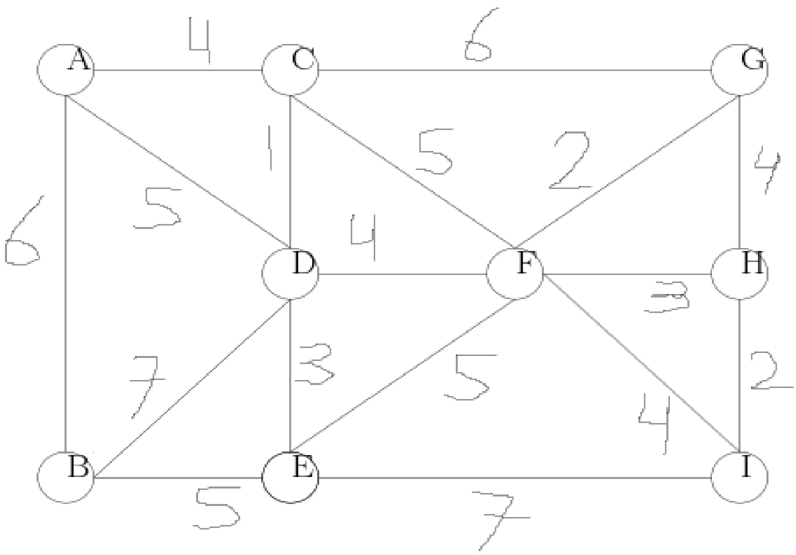
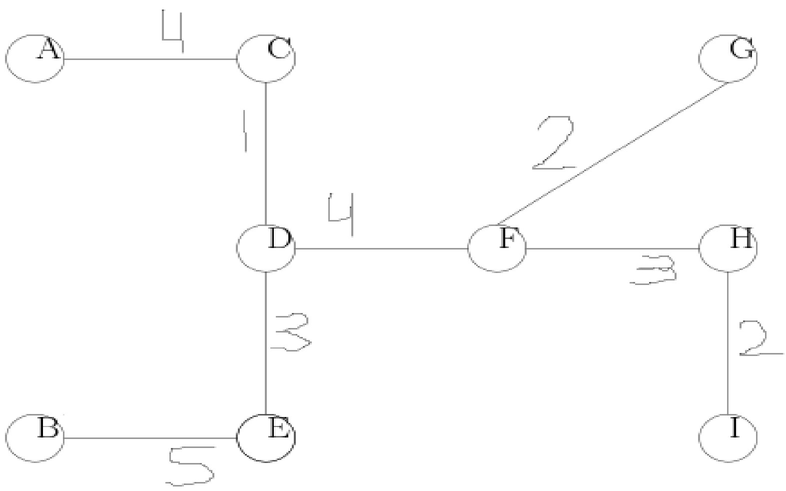


Figure 1: Exercise 4

Exercise 4

(a) **[10 points]** Consider the problem of finding a minimum weight spanning tree in the non-directed graph of Figure 1 (where edge weights have been indicated in the figure) using Kruskal's algorithm. Make clear in which **order** the edges are picked by the algorithm and draw the minimum weight spanning tree which is finally obtained. Also explicitly write down the instructions of Kruskal's algorithm which prescribe the order in which edges to be included in the minimum spanning tree are picked.

Solution exercise 4: Kruskal's algorithm orders edges from lowest weight to highest weight and then picks edges in that order under the condition that the next edge on the list is only added if there is no cycle formed by adding that edge. Applying this algorithm a possible order for edges to be added is: $\{C, D\}, \{F, G\}, \{H, I\}, \{F, H\}, \{D, E\}, \{A, C\}, \{D, F\}, \{B, E\}$.
The resulting minimum spanning tree is then as follows which has total weight of 24:



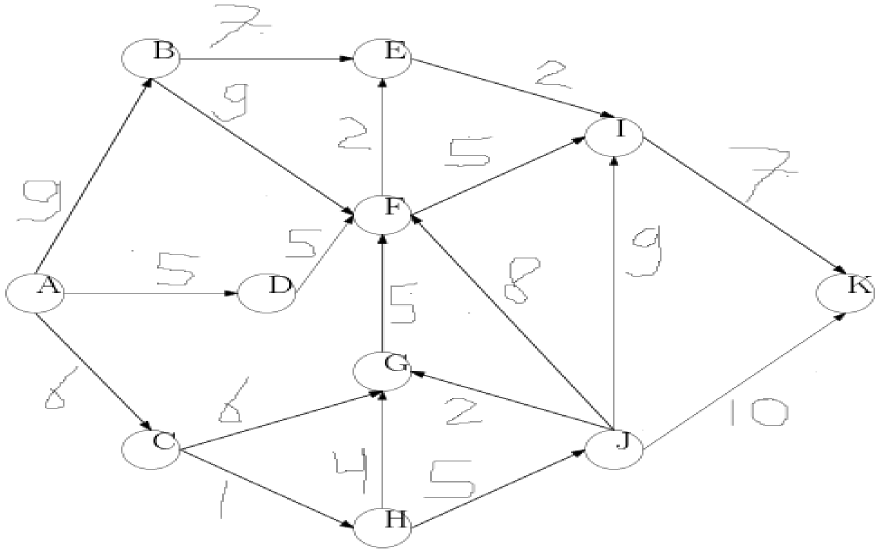


Figure 2: Exercise 5

Exercise 5

Consider the directed graph $G = (V, A)$ as drawn in Figure 2 with lengths of the arcs as indicated in the figure.

(a) **[10 points]** Apply Dijkstra's algorithm to determine the **shortest path** from node A to node K in this directed graph. It should be clear from the computations you write down that you have applied Dijkstra's algorithm to find the shortest path. After finishing the computations also indicate explicitly the shortest path which is obtained and the length of that shortest path.

(b) **[10 points]** Apply backward recursion to determine the **longest path** from node A to node K in the directed graph. Make sure it is clear in which order you calculate the function values and explain why the function values are calculated in that order.
Besides the function values calculated by applying backward recursion also indicate explicitly the longest path which is obtained and the length of that longest path.

Solution exercise 5:

(a) The calculations from Dijkstra's algorithm are summarized in the following table where label (x, y) for a node means that x is the shortest distance from A to the node obtained until the current iteration and y is the predecessor of the node on that shortest path to the node. Once the label of a node has become permanent it is no longer written in this table in the following iterations (since it will not change anymore). If the label is followed by * it means that this label becomes permanent in the current iteration.

iteration	B	C	D	E	F	G	H	I	J	K
1	$(9, A)$	$(6, A)$	$(5, A)^*$							
2	$(9, A)$	$(6, A)^*$			$(10, D)$					
3	$(9, A)$				$(10, D)$	$(12, C)$	$(7, C)^*$			
4	$(9, A)^*$				$(10, D)$	$(11, H)$			$(12, H)$	
5				$(16, B)$	$(10, D)^*$	$(11, H)$			$(12, H)$	
6				$(12, F)$		$(11, H)^*$		$(15, F)$	$(12, H)$	
7				$(12, F)^*$				$(15, F)$	$(12, H)$	
8								$(14, E)$	$(12, H)^*$	
9								$(14, E)^*$		$(22, J)$
10										$(21, I)^*$

Backtracking from node K we obtain that the corresponding shortest path is: $A \to D \to F \to E \to I \to K$ having total length 21.

(b) In the acyclic directed graph we first number the nodes such that for every arc (i, j) in the graph it holds that $i < j$. Such numbering is for example $A = 1$, $B = 2$, $D = 3$, $C = 4$, $H = 5$, $J = 6$, $G = 7$, $F = 8$, $E = 9$, $I = 10$, $K = 11$.

Now define the value function $f(i)$ to be the length of the longest path from node numbered i to destination node $K = 11$. Initialize $f(K) = f(11) = 0$ and compute the other function values by the backward recursion $f(i) = \max_{j:(i,j)\in A}[w(i, j) + f(j)]$.
Then it follows consecutively (doing calculations in reverse order of the numbering of the nodes) that $f(I) = 7$, $f(E) = 9$, $f(F) = 12$, $f(G) = 17$, $f(J) = 20$, $f(H) = 25$, $f(C) = 26$, $f(D) = 17$, $f(B) = 21$, $f(A) = 32$.
Backtracking we obtain as longest path the path $A \to C \to H \to J \to F \to I \to K$. It is easily seen that the length of this path is indeed 32 corresponding with the calculated function value $f(A)$.