Exam Operations Research

Date: March 26, 2021
Time: 12:15 - 14:30

Points per exercise:

- Exercise 1 has a total of 30 points (a(10), b(5), c(5) and d(10)).
- Exercise 2 has a total of 15 points.
- Exercise 3 has a total of 15 points (a(5) and b(10)).
- Exercise 4 has a total of 20 points (a(10) and b(10)).
- Exercise 5 has a total of 10 points.

Thus in total 90 points can be obtained. The exam grade is determined as follows:

$$\text{Exam grade } = 1 + \frac{\text{total number of obtained points}}{10}.$$

- Calculator is allowed.
- This exam consists of 5 pages, including this one.
- The duration of this exam is **2 hours and 15 minutes**. When you finish working you notify the host of your meeting. Within 15 minutes after you finish working you have to upload the scanned pdf file in Canvas in the assignment for the exam. After uploading the pdf file you notify the host again.
- Students who have obtained permission for extra time may use an *additional 30 minutes*.

Exercise 1

Consider the following LP which is referred to as the "primal LP".

$$\begin{aligned} \max \quad & z = 2x_1 - x_2 + 2x_3 + 3x_4 \\ \text{s.t.} \quad & x_1 - 2x_2 + 2x_4 \leq 16 \\ & 3x_1 + 2x_2 + x_3 + x_4 = 18 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(a) **[10 points]** Determine the dual of this LP.

(b) **[5 points]** The primal LP has been solved by the simplex method resulting in the following final tableau where s_1 is the slack variable introduced in the first constraint and r_1 is the artificial variable introduced in the second constraint:

Basic	z	x_1	x_2	x_3	x_4	s_1	r_1	value
z	1	4.50	4	0	0	0.50	2	44
x_4	0	0.50	−1	0	1	0.50	0	8
x_3	0	2.50	3	1	0	−0.50	1	10

Determine the optimal solution of the primal LP and optimal objective value. Determine also the optimal solution of the dual LP and optimal dual objective function.

(c) **[5 points]** Suppose the right hand sides of both constraints are changed to 20 (instead of respectively 16 and 18). Determine the optimal solution and optimal objective value of this modified LP.

(d) **[10 points]** Now instead of changing the right hand sides of the constraints the extra constraint $x_3 \leq 7$ is added to the primal LP. Apply the dual simplex method to determine the optimal solution and optimal objective value of this modified LP.
Instruction: If you do the calculations correctly you get some fractions in the simplex tableau, but all these fractions are of the form $\frac{1}{6}k$ *with* k *integer. Moreover, it will require only one pivot step to find the new optimal solution.*

Exercise 2

A certain area has 4 major towns. Health coordination has to choose locations in this area where people from this area can be vaccinated against the coronavirus. In this area there are 5 locations which could be established as a vaccination centre. The cost of establishing a vaccination centre at location i is C_i for $i = 1, 2, \ldots, 5$. A restriction in choosing the vaccination locations is that for each of the 4 major towns there should be at least one vaccination centre established within a distance of at most 10 km from that town. In the provided table you find all the distances in kilometers between each of the 4 towns and the 5 possible locations for a vaccination centre. Another restriction is that it is already decided that either location 2 or location 4 will be established as vaccination centre but not both locations. The objective is to establish vaccination centres for a minimal total cost satisfying all restrictions.

Location\Town	1	2	3	4
1	12	4	9	16
2	14	8	12	6
3	10	15	11	7
4	7	12	5	16
5	11	15	7	8

(a) **[15 points]**

Formulate the problem of minimizing the total costs for establishing vaccination centres under the given restrictions as an integer linear program (ILP). Explain all variables and constraints in your ILP formulation of this problem.

Exercise 3

Consider the following ILP with four binary decision variables x_1, x_2, x_3, x_4.

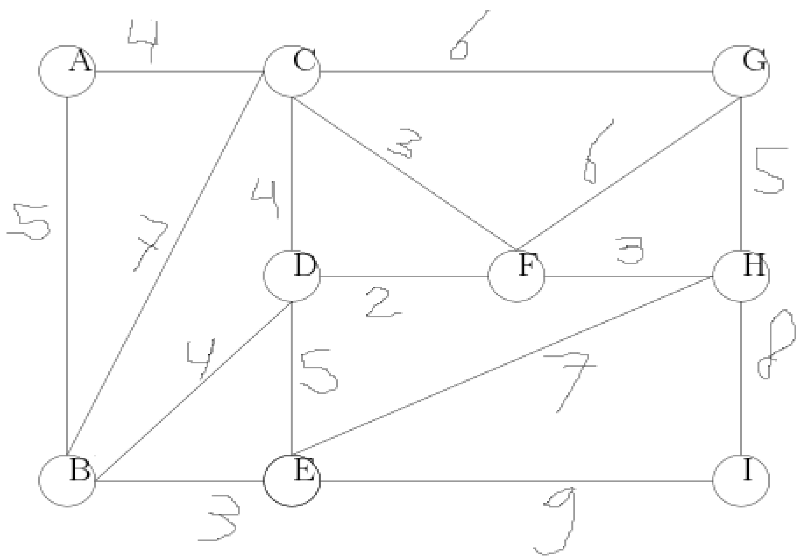
$$\begin{aligned} \max \quad & z = x_1 + 4x_2 + 6x_3 + 3x_4 \\ \text{s.t.} \quad & 3x_1 + 11x_2 + 17x_3 + 8x_4 \leq 23 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

(a) **[5 points]** Write down the LP relaxation of this ILP and determine the unique optimal solution of the LP relaxation. Also determine the corresponding optimal objective value of the LP relaxation.
Instruction: The special form of the ILP makes it possible to quickly determine the optimal solution of the LP relaxation. It is not needed to apply the simplex method.

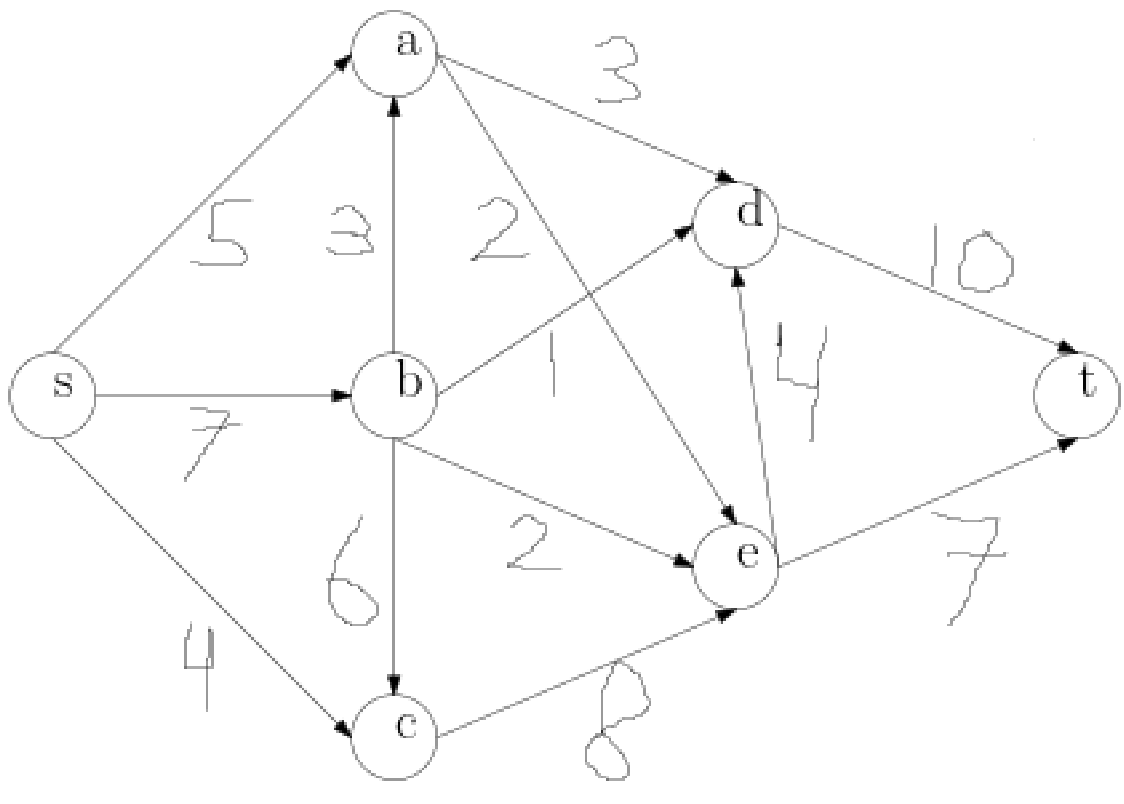
(b) **[10 points]** Solve the ILP by applying the **branch-and-bound** method. Clearly indicate in which order you compute the nodes of the search tree and where you prune the search tree. Also indicate which pruning criterion you use when you prune.

Exercise 4

(a) **[10 points]** Consider the problem of finding a minimum weight spanning tree in the non-directed graph below (where edge weights have been indicated) using Prim's algorithm starting from the tree containing only node A. Make clear in which **order** the edges are picked by the algorithm and draw the minimum weight spanning tree which is finally obtained. Explain the order in which edges to be included in the minimum spanning tree are picked by this algorithm.

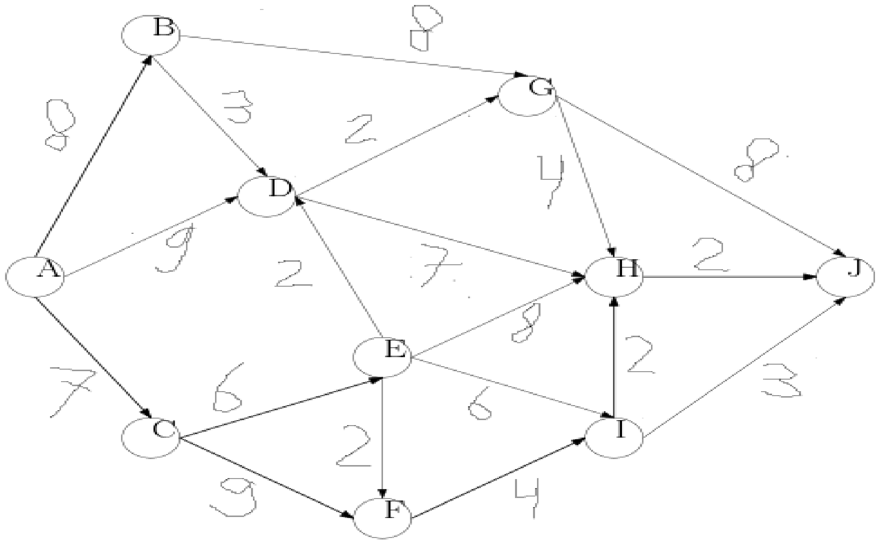


(b) **[10 points]** Consider the instance of the maximum flow problem shown in the directed graph below where the arc capacities are indicated by the numbers near the arcs.
Let the current flow f (which is feasible but not maximal) be as follows: $f_{sa} = 3$, $f_{sb} = 7$, $f_{sc} = 4$, $f_{ad} = 3$, $f_{ae} = 2$, $f_{ba} = 2$, $f_{bd} = 1$, $f_{bc} = 2$, $f_{be} = 2$, $f_{ce} = 6$, $f_{dt} = 7$, $f_{ed} = 3$, $f_{et} = 7$ and no flow on other arcs. Draw the residual graph D^f corresponding to this flow f. Continue from this residual graph the Ford-Fulkerson algorithm to determine a maximum flow from s to t. State the value of the flow and show that it is maximal by giving a minimum s-t cut of that value.



Exercise 5

Consider the acyclic directed graph shown below with lengths of the arcs as indicated in the graph.



(a) **[10 points]**. Apply dynamic programming to determine the **longest path** from node A to node J in this directed graph. Define an appropriate value function and use backward recursion to compute for all states the function value. Determine the length of the longest path and make clear which path is the longest path you have obtained.