Exam Operations Research

Date: March 26, 2021
Time: 12:15 - 14:30

Points per exercise:

- Exercise 1 has a total of 30 points (a(10), b(5), c(5) and d(10)).
- Exercise 2 has a total of 15 points.
- Exercise 3 has a total of 15 points (a(5) and b(10)).
- Exercise 4 has a total of 20 points (a(10) and b(10)).
- Exercise 5 has a total of 10 points.

Thus in total 90 points can be obtained. The exam grade is determined as follows:

$$\text{Exam grade } = 1 + \frac{\text{total number of obtained points}}{10}.$$

- Calculator is allowed.
- This exam consists of 5 pages, including this one.
- The duration of this exam is **2 hours and 15 minutes**. When you finish working you notify the host of your meeting. Within 15 minutes after you finish working you have to upload the scanned pdf file in Canvas in the assignment for the exam. After uploading the pdf file you notify the host again.
- Students who have obtained permission for extra time may use an *additional 30 minutes*.

Exercise 1

Consider the following LP which is referred to as the "primal LP".

$$\begin{aligned} \max \quad & z = 2x_1 - x_2 + 2x_3 + 3x_4 \\ \text{s.t.} \quad & x_1 - 2x_2 + 2x_4 \leq 16 \\ & 3x_1 + 2x_2 + x_3 + x_4 = 18 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(a) **[10 points]** Determine the dual of this LP.

(b) **[5 points]** The primal LP has been solved by the simplex method resulting in the following final tableau where s_1 is the slack variable introduced in the first constraint and r_1 is the artificial variable introduced in the second constraint:

Basic	z	x_1	x_2	x_3	x_4	s_1	r_1	value
z	1	4.50	4	0	0	0.50	2	44
x_4	0	0.50	−1	0	1	0.50	0	8
x_3	0	2.50	3	1	0	−0.50	1	10

Determine the optimal solution of the primal LP and optimal objective value. Determine also the optimal solution of the dual LP and optimal dual objective function.

(c) **[5 points]** Suppose the right hand sides of both constraints are changed to 20 (instead of respectively 16 and 18). Determine the optimal solution and optimal objective value of this modified LP.

(d) **[10 points]** Now instead of changing the right hand sides of the constraints the extra constraint $x_3 \leq 7$ is added to the primal LP. Apply the dual simplex method to determine the optimal solution and optimal objective value of this modified LP.
Instruction: If you do the calculations correctly you get some fractions in the simplex tableau, but all these fractions are of the form $\frac{1}{6}k$ *with k integer. Moreover, it will require only one pivot step to find the new optimal solution.*

Solution exercise 1:

(a) The dual LP is:

$$\begin{aligned} \min \quad & z = 16y_1 + 18y_2 \\ \text{s.t.} \quad & y_1 + 3y_2 \geq 2 \\ & -2y_1 + 2y_2 \geq -1 \\ & y_2 \geq 2 \\ & 2y_1 + y_2 \geq 3 \\ & y_1 \geq 0, y_2 \text{ unrestricted} \end{aligned}$$

(b) The optimal solution of the primal LP is $x_1^* = 0$, $x_2^* = 0$, $x_3^* = 10$, $x_4^* = 8$ and optimal objective value $z^* = 44$. The optimal solution of the dual LP is $y_1^* = 0.50$, $y_2^* = 2$ with optimal dual objective value $w^* = 44$.

(c) The change in the right hand side of the first constraint is $\Delta_1 = 20 - 16 = 4$ and in the second constraint is $\Delta_2 = 20 - 18 = 2$. Thus as corresponding basic solution of the modified LP we obtain $x_4^* = 8 + 0.50\Delta_1 + 0\Delta_2 = 10$ and $x_3^* = 10 - 0.50\Delta_1 + 1\Delta_2 = 10$ and for the nonbasic variables we still have $x_1^* = x_2^* = 0$. Notice that this solution is feasible and thus optimal for the modified LP. It follows that the new optimal objective value is $z^* = 44 + 0.50\Delta_1 + 2\Delta_2 = 50$.

(d) Rewriting the new constraint $x_3 \leq 7$ to $x_3 + s_2 = 7$ with s_2 slack variable and then adding the corresponding s_2- row to the final simplex tableau of the primal LP gives the following tableau:

Basic	z	x_1	x_2	x_3	x_4	s_1	r_1	s_2	value
z	1	4.50	4	0	0	0.50	2	0	44
x_4	0	0.50	-1	0	1	0.50	0	0	8
x_3	0	2.50	3	1	0	-0.50	1	0	10
s_2	0	0	0	1	0	0	0	1	7

Subtracting the x_3 row from the s_2 row to get a zero in at the intersection of the x_3-column and s_2-row we obtain:

Basic	z	x_1	x_2	x_3	x_4	s_1	r_1	s_2	value
z	1	4.50	4	0	0	0.50	2	0	44
x_4	0	0.50	-1	0	1	0.50	0	0	8
x_3	0	2.50	3	1	0	-0.50	1	0	10
s_2	0	-2.50	-3	0	0	0.50	-1	1	-3

Now the value of s_2 has become negative and thus s_2 has to leave the basis. By the minimal ratio rule for the dual simplex method it follows that x_2 will enter the basis instead of s_2. Performing the corresponding pivot step we obtain the following tableau:

Basic	z	x_1	x_2	x_3	x_4	s_1	r_1	s_2	value
z	1	$\frac{7}{6}$	0	0	0	$\frac{7}{6}$	$\frac{2}{3}$	$\frac{4}{3}$	40
x_4	0	$\frac{4}{3}$	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	9
x_3	0	0	0	1	0	0	0	1	7
x_2	0	$\frac{5}{6}$	1	0	0	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$	1

From this tableau it can be concluded that the optimal solution of the modified LP is $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 7$, $x_4^* = 9$ with optimal objective value $z^* = 40$.

Exercise 2

A certain area has 4 major towns. Health coordination has to choose locations in this area where people from this area can be vaccinated against the coronavirus. In this area there are 5 locations which could be established as a vaccination centre. The cost of establishing a vaccination centre at location i is C_i for $i = 1, 2, \ldots, 5$. A restriction in choosing the vaccination locations is that for each of the 4 major towns there should be at least one vaccination centre established within a distance of at most 10 km from that town. In the provided table you find all the distances in kilometers between each of the 4 towns and the 5 possible locations for a vaccination centre. Another restriction is that it is already decided that either location 2 or location 4 will be established as vaccination centre but not both these locations. The objective is to establish vaccination centres for a minimal total cost satisfying all restrictions.

Location\Town	1	2	3	4
1	12	4	9	16
2	14	8	12	6
3	10	15	11	7
4	7	12	5	16
5	11	15	7	8

(a) **[15 points]** Formulate the problem of minimizing the total costs for establishing vaccination centres under the given restrictions as an integer linear program (ILP). Explain all variables and constraints in your ILP formulation of this problem.

Solution exercise 2: Define binary variables x_i for $i = 1, 2, \ldots, 5$ where $x_i = 1$ if location i is established as vaccination centre. Using these decision variables the problem can be formulated as the following ILP:

$$
\begin{aligned}
\min \quad & w = \sum_{i=1}^{5} C_i x_i & \\
\text{s.t.} \quad & x_3 + x_4 \geq 1 & \text{town 1 distance restriction} \\
& x_1 + x_2 \geq 1 & \text{town 2 distance restriction} \\
& x_1 + x_4 + x_5 \geq 1 & \text{town 3 distance restriction} \\
& x_2 + x_3 + x_5 \geq 1 & \text{town 4 distance restriction} \\
& x_2 + x_4 = 1 & \text{vaccination centre in either location 2 or location 4} \\
& \text{All variables } x_i \in \{0, 1\} &
\end{aligned}
$$

Exercise 3

Consider the following ILP with four binary decision variables x_1, x_2, x_3, x_4.

$$\begin{array}{ll} \max & z = x_1 + 4x_2 + 6x_3 + 3x_4 \\ \text{s.t.} & 3x_1 + 11x_2 + 17x_3 + 8x_4 \leq 23 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{array}$$

(a) **[5 points]** Write down the LP relaxation of this ILP and determine the unique optimal solution of the LP relaxation. Also determine the corresponding optimal objective value of the LP relaxation.
Instruction: The special form of the ILP makes it possible to quickly determine the optimal solution of the LP relaxation. It is not needed to apply the simplex method.

(b) **[10 points]** Solve the ILP by applying the **branch-and-bound** method. Clearly indicate in which order you compute the nodes of the search tree and where you prune the search tree. Also indicate which pruning criterion you use when you prune.

Solution exercise 3:

(a) The ILP has the following LP relaxation:

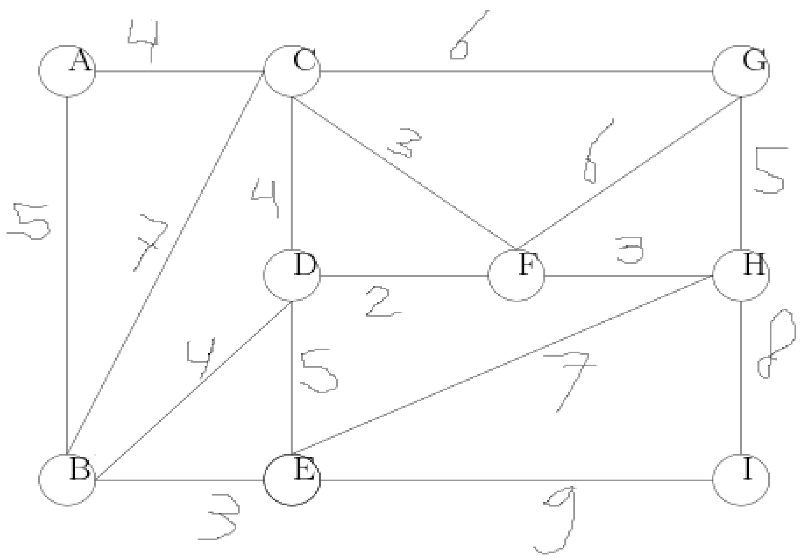
$$\begin{array}{ll} \max & z = x_1 + 4x_2 + 6x_3 + 3x_4 \\ \text{s.t.} & 3x_1 + 11x_2 + 17x_3 + 8x_4 \leq 23 \\ & 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3, 4 \end{array}$$

Notice that the problem is an $0-1$ knapsack problem and thus the optimal solution of the LP relaxation can be found by ordering the four variables according to value-weight ratio and then applying the greedy algorithm. Since $\frac{3}{8} > \frac{4}{11} > \frac{6}{17} > \frac{1}{3}$ the greedy algorithm gives as optimal solution of the LP relaxation $x_4 = 1$, $x_2 = 1$, $x_3 = \frac{4}{17}$, $x_1 = 0$ with corresponding optimal objective value $z^* = 8\frac{7}{17}$.

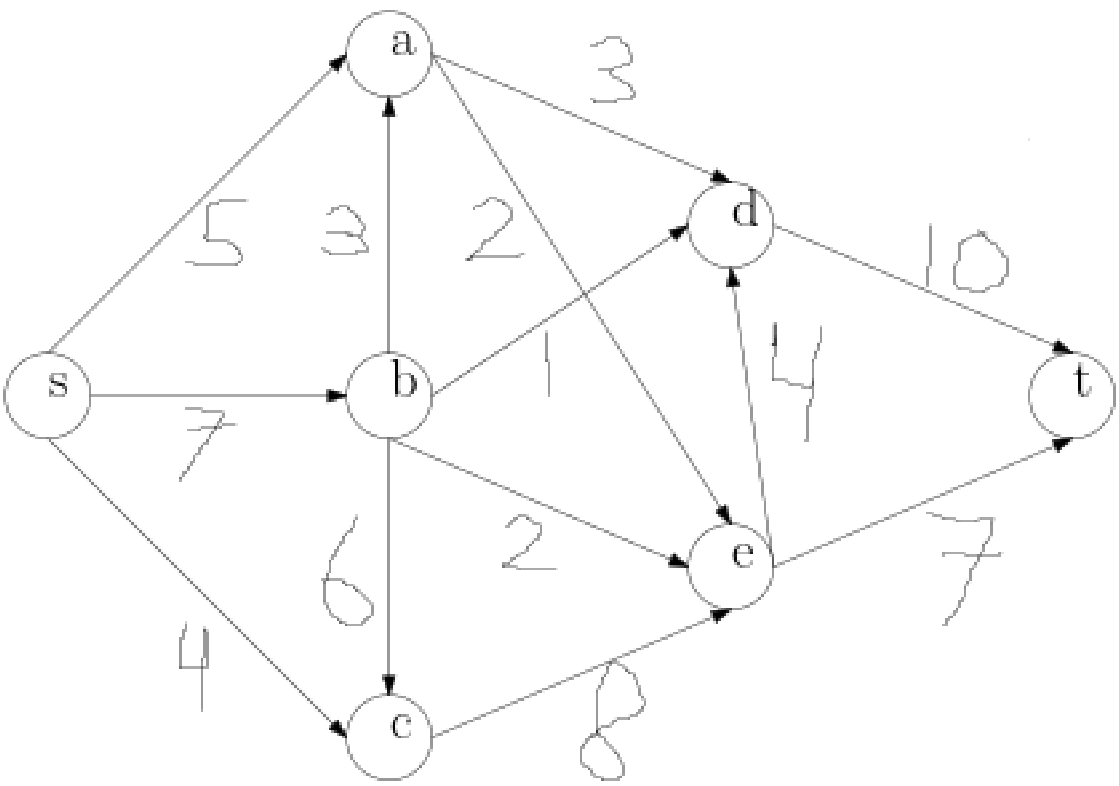
(b) From (a) we already have for starting node 0 the upperbound $z^0 = 8\frac{7}{17}$. From starting node 0 we have to branch on variable x_3.
Put $x_3 = 0$ for subproblem 1 (corresponding to node 1 in the search tree). Then the LP relaxation has optimal solution $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1$ with value $z^1 = 8$. This solution is feasible for the original ILP and thus the objective value 8 is a lowerbound for the original ILP. Moreover, for node 1 an optimal solution has been obtained and we do not have to branch from node 1.
Put $x_3 = 1$ for subproblem 2 (corresponding to node 2 in the search tree). The LP relaxation in node 2 has optimal solution $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = \frac{3}{4}$ with value $z^2 = 8\frac{1}{4}$. Because integer solutions of this problem can clearly have only integral values it follows that from node 2 never a feasible solution with value more than 8 can be obtained (in fact that there is no feasible solution of value more than 8 could already be concluded from $z^0 = 8\frac{7}{17}$ in node 0). Thus from node 2 never a solution can be obtained that improves on the feasible solution having value 8 we have already found in node 1.
Thus we prune in node 2 and conclude that $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1$ is an optimal solution of the ILP and the optimal objective value of the ILP is $z^* = 8$.

Exercise 4

(a) **[10 points]** Consider the problem of finding a minimum weight spanning tree in the non-directed graph below (where edge weights have been indicated) using Prim's algorithm starting from the tree containing only node A. Make clear in which **order** the edges are picked by the algorithm and draw the minimum weight spanning tree which is finally obtained. Explain the order in which edges to be included in the minimum spanning tree are picked by this algorithm.

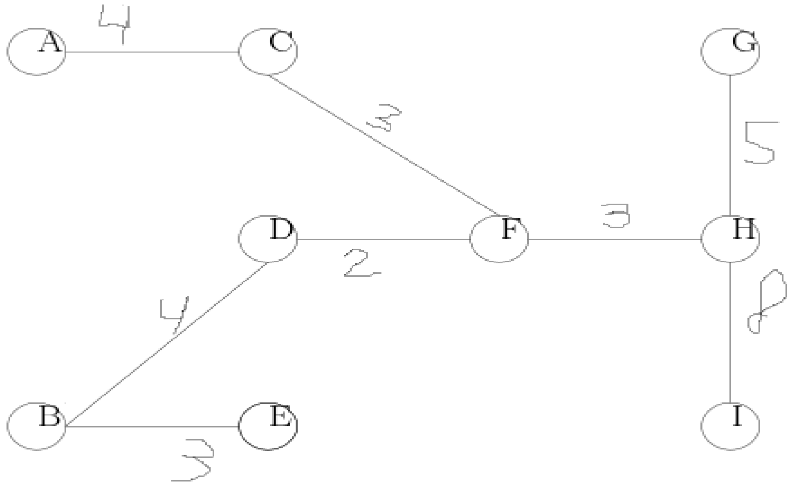


(b) **[10 points]** Consider the instance of the maximum flow problem shown in the directed graph below where the arc capacities are indicated by the numbers near the arcs.
Let the current flow f (which is feasible but not maximal) be as follows: $f_{sa} = 3$, $f_{sb} = 7$, $f_{sc} = 4$, $f_{ad} = 3$, $f_{ae} = 2$, $f_{ba} = 2$, $f_{bd} = 1$, $f_{bc} = 2$, $f_{be} = 2$, $f_{ce} = 6$, $f_{dt} = 7$, $f_{ed} = 3$, $f_{et} = 7$ and no flow on other arcs. Draw the residual graph D^f corresponding to this flow f. Continue from this residual graph the Ford-Fulkerson algorithm to determine a maximum flow from s to t. State the value of the flow and show that it is maximal by giving a minimum s-t cut of that value.

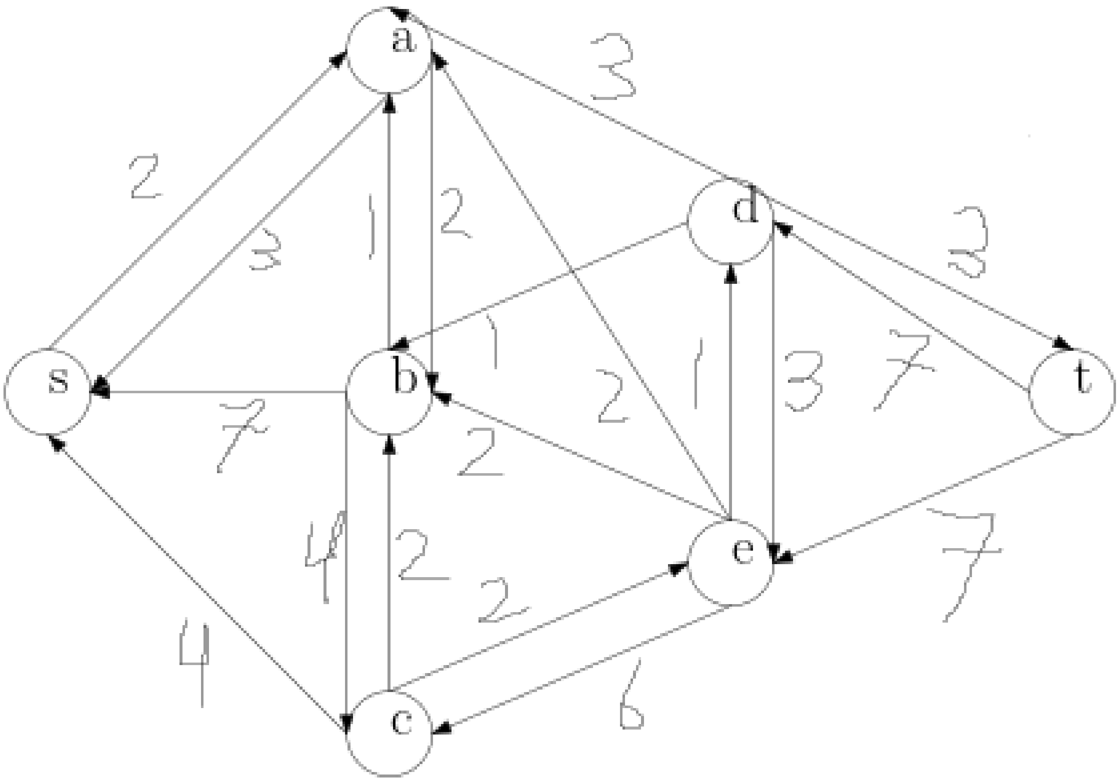


Solution exercise 4:

(a) Prim's algorithm adds an edge of lowest weight under the condition that the edge should be connected to the existing tree and there is no cycle formed by adding that edge. Applying this algorithm the following edges are added in the following order: $\{A, C\}, \{C, F\}, \{D, F\}, \{F, H\}, \{B, D\}, \{B, E\}, \{G, H\}, \{H, I\}$. The resulting minimum spanning tree is then as follows:



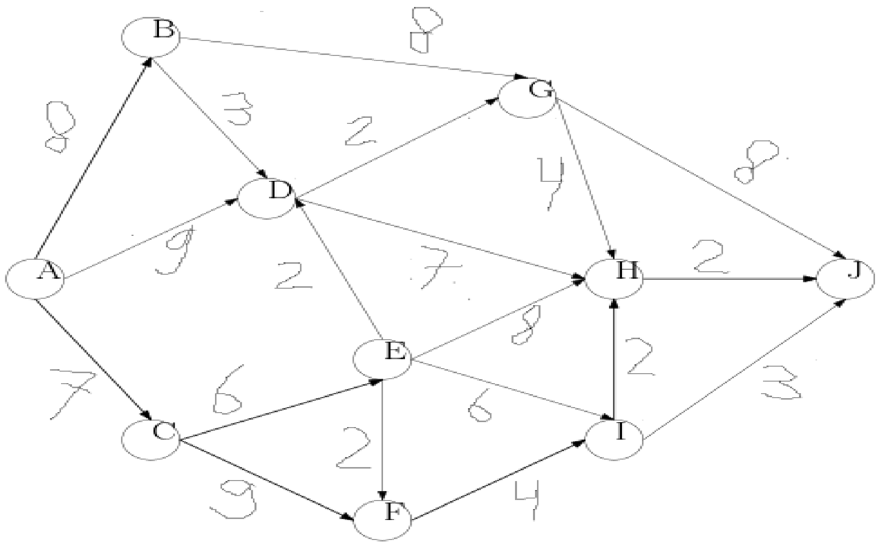
(b) The residual graph for the given flow is as follows:



An augmenting path in the above residual graph is: $s \to a \to b \to c \to e \to d \to t$ on which an extra flow of maximal 1 can be pushed. Pushing this extra flow of 1 will remove the arc from e to d in the next residual graph. It is easily seen that in that next residual graph there will be no longer a path from s to t because nodes d and t can not be reached from s. Thus the resulting flow after pushing 1 on the augmenting path should be maximal. This resulting flow is: $f_{sa} = 4$, $f_{sb} = 7$, $f_{sc} = 4$, $f_{ad} = 3$, $f_{ae} = 2$, $f_{ba} = 1$, $f_{bd} = 1$, $f_{bc} = 3$, $f_{be} = 2$, $f_{ce} = 7$, $f_{dt} = 8$, $f_{ed} = 4$, $f_{et} = 7$ having value 15. To show that this is indeed a maximal flow an $s-t$ cut in the original graph of the same total capacity of 15 should be provided. This is the cut consisting of the arcs $\{(a,d),(b,d),(e,d),(e,t)\}$ which indeed has a total capacity of $3+1+4+7=15$. This minimum cut is the cut between the nodes $\{s,a,b,c,e\}$ and the nodes $\{d,t\}$.

Exercise 5

Consider the acyclic directed graph shown below with lengths of the arcs as indicated in the graph.



(a) **[10 points]**. Apply dynamic programming to determine the **longest path** from node A to node J in this directed graph. Define an appropriate value function and use backward recursion to compute for all states the function value. Determine the length of the longest path and make clear which path is the longest path you have obtained.

Solution exercise 5:
Because the directed graph is acyclic we can number the nodes in the graph such that there are only forward arcs with respect to that numbering. After such numbering of the nodes backward recursion can be applied using the numbering of the nodes. Such a numbering which is applicable for backward recursion is $A=1$, $B=2$, $C=3$, $E=4$, $D=5$, $F=6$, $G=7$, $I=8$, $H=9$, $J=10$.
Define the value function $f(i)$ to be the length of the longest path from node numbered i to destination node $J=10$. Initialize $f(10)=0$ and compute the other function values in backward order by the recursion $f(i)=\max_{j:(i,j)\in A}[w(i,j)+f(j)]$. Then it follows consecutively (doing calculations in reverse order of the numbering of the nodes) that $f(9)=2$, $f(8)=4$, $f(7)=8$, $f(6)=8$, $f(5)=10$, $f(4)=12$, $f(3)=18$, $f(2)=16$, $f(1)=25$.
Backtracking we obtain as longest path the path $A\rightarrow C\rightarrow E\rightarrow D\rightarrow G\rightarrow J$. It is easily checked that the length of this path is indeed 25 corresponding with the function value $f(1)$.