Exam Operations Research

Date: July 10, 2020
Time: 10:00 - 12:15

Points per exercise:

- Exercise 1 has a total of 30 points (a(10), b(5) , c(5) and d(10)).
- Exercise 2 has a total of 15 points.
- Exercise 3 has a total of 20 points (a(5), b(5) and c(10)).
- Exercise 4 has a total of 10 points.
- Exercise 5 has a total of 15 points (a(10) and b(5)).

Thus in total 90 points can be obtained. The exam grade is determined as follows:

$$\text{Exam grade } = 1 + \frac{\text{total number of obtained points}}{10}.$$

- Calculator is allowed.
- This exam consists of 4 pages, including this one.
- The duration of this exam is **2 hours and 15 minutes**. When you finish working you notify the host of your meeting. Within 15 minutes after you finish working you have to upload the scanned pdf file in Canvas in the assignment for the July 10 resit exam. After uploading the pdf file you notify the host again.
- Students who have obtained permission for extra time may use an *additional 30 minutes*.

Exercise 1

Consider the following LP which is referred to as the "primal LP".

$$\begin{aligned} \min \quad & w = 4x_1 - x_2 + 3x_3 + 2x_4 \\ \text{s.t.} \quad & x_1 - x_2 + x_3 + 3x_4 \geq 20 \\ & 2x_1 + x_2 + x_3 + x_4 = 16 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(a) **[10 points]** Determine the dual of this LP.

(b) **[5 points]** The primal LP has been solved by the simplex method resulting in the following final simplex tableau where s_1 is the surplus variable from the first constraint, r_1 is the artificial variable from the first constraint and r_2 is the artificial variable from the second constraint:

Basic	w	x_1	x_2	x_3	x_4	s_1	r_1	r_2	value
w	1	-3.75	0	-2.50	0	-0.75	0.75	-0.25	11.00
x_4	0	0.75	0	0.50	1.00	-0.25	0.25	0.25	9.00
x_2	0	1.25	1.00	0.50	0	0.25	-0.25	0.75	7.00

Determine the optimal solution of the primal LP and optimal objective value. Determine also the optimal solution of the dual LP and optimal dual objective function.

(c) **[5 points]** The right hand side of the second constraint is changed to 12 instead of 16. Determine the optimal solution of this modified primal LP and the corresponding optimal objective value.
Instruction: This optimal solution can be found by sensitivity analysis using the above simplex tableau without doing any pivot step.

(d) **[10 points]** Now the right hand side of the second contraint is not changed (thus it is 16 as in the original LP problem), but the coefficent in the objective function before x_4 is changed to -2 instead of 2. Adjust the simplex tableau accordingly and apply the simplex method to determine the optimal solution and optimal objective value of this modified LP. It will require one pivot step to find the new optimal solution. Also explain why it follows from the simplex tableau which you obtain after one pivot step that the corresponding solution is the optimal solution of the modified LP.
Instruction: Since the primal solution you have determined in exercise 1b is still feasible (but it will be no longer optimal) you can leave out the columns of the two artificial variables in your calculations to spare some time. Moreover, if you do the calculations correctly then in the final simplex tableau all values will be integers.

Solution exercise 1:

(a) The dual LP is:

$$\begin{aligned} \max \quad & z = 20y_1 + 16y_2 \\ \text{s.t.} \quad & y_1 + 2y_2 \leq 4 \\ & -y_1 + y_2 \leq -1 \\ & y_1 + y_2 \leq 3 \\ & 3y_1 + y_2 \leq 2 \\ & y_1 \geq 0, y_2 \text{ unrestricted} \end{aligned}$$

(b) The optimal solution of the primal LP is $x_1^* = 0$, $x_2^* = 7$, $x_3^* = 0$, $x_4^* = 9$ and optimal objective value $w^* = 11$. The optimal solution of the dual LP is $y_1^* = 0.75$, $y_2^* = -0.25$ with optimal dual objective value $z^* = 11$.

(c) The change in the right hand side of the second constraint is $\Delta_2 = 12 - 16 = -4$. The new optimal solution will have the same basic variables since no pivot step is needed.
In this new optimal solution $x_4^* = 9 + 0.25\Delta_2 = 8$ and $x_2^* = 7 + 0.75\Delta_2 = 4$ and for the nonbasic variables we still have $x_1^* = x_3^* = 0$. The new optimal objective value is $w^* = 11 + (-0.25)\Delta_2 = 12$.

(d) The modified simplex tableau omitting the columns of the artificial variables is:

Basic	w	x_1	x_2	x_3	x_4	s_1	value
w	1	−3.75	0	−2.50	4.00	−0.75	11.00
x_4	0	0.75	0	0.50	1.00	−0.25	9.00
x_2	0	1.25	1.00	0.50	0	0.25	7.00

Subtracting 4 times the x_4 row from the w row to get a zero coefficient in the objective row for the nonbasic variable x_4 we obtain:

Basic	w	x_1	x_2	x_3	x_4	s_1	value
w	1	−6.75	0	−4.50	0	0.25	−25.00
x_4	0	0.75	0	0.50	1.00	−0.25	9.00
x_2	0	1.25	1.00	0.50	0	0.25	7.00

Now the coefficient before s_1 in the objective row has become positive. So for the pivot step s_1 will enter the basis and then x_2 (having positive coefficient in the s_1 column) will leave the basis. Then we obtain the following tableau:

Basic	w	x_1	x_2	x_3	x_4	s_1	value
w	1	−8.00	−1.00	−5.00	0	0	−32.00
x_4	0	2.00	1.00	1.00	1.00	0	16.00
s_1	0	5.00	4.00	2.00	0	1.00	28.00

From this tableau it follows that the optimal solution of the modified LP is $x_1^* = x_2^* = x_3^* = 0$, $x_4^* = 16$ with optimal objective value $w^* = -32$.

Exercise 2

There are 7 files ($i = 1, 2, \ldots, 7$) which have to be stored on USB sticks. For storing these files there are two USB sticks available with storage capacities of respectively 32 MB and 16 MB. The sizes of the 7 files are respectively 6 MB, 5 MB, 3 MB, 9 MB, 4 MB, 6 MB and 11 MB. Each file should be stored at one of the USB sticks. Moreover, because of a security issue the files numbered 2 and 7 may not be stored on the same USB stick.

(a) **[15 points]** Formulate an integer linear program for this problem. Formulate the objective function such that if there exist feasible solution(s) that then the total amount of MB's which are stored on the stick with 32 MB capacity is minimized. Explain the meaning of all variables and constraints you have in your formulation. Pay attention that the formulation (besides that you may use integer or binary variables) has to be linear.

Solution exercise 2:
Define binary variables x_i for $i = 1, 2, \ldots, 7$ where $x_i = 1$ if file i is stored on the 32 MB stick and $x_i = 0$ if file i is stored on the 16 MB stick. Let y_1 be the total amount of MB's stored on the 32 MB stick and let y_2 be the total amount of MB's stored on the 16 MB stick. Using these decision variables we can formulate the problem as follows:

$$
\begin{aligned}
\min \quad & w = y_1 \\
\text{s.t.} \quad & y_1 = 6x_1 + 5x_2 + 3x_3 + 9x_4 + 4x_5 + 6x_6 + 11x_7 \\
& y_1 + y_2 = 44 \text{ (this is the total amount of MB's of all 7 files)} \\
& y_1 \leq 32 \\
& y_2 \leq 16 \\
& x_2 + x_7 = 1 \text{ (this constraint to make sure that file 2 and file 7 are stored on different sticks)} \\
& \text{All variables } x_i \in \{0, 1\}, y_1 \geq 0, y_2 \geq 0
\end{aligned}
$$

Exercise 3

Consider the following problem. You are packing your suitcase for a journey and need to decide what items to bring. You have 4 items (just one of each); each has a certain value to you, and a certain weight:

weight (kg)	4	7	3	5
value	10	17	8	14

You do not want to have more than 10 kg of weight in the suitcase. The problem is to choose under that weight restriction which items should be put into the suitcase maximizing the total value of those items.

(a) **[5 points]** Formulate an integer linear program for this problem.

(b) **[5 points]** Determine the unique optimal solution of the LP relaxation of this problem. Also determine the corresponding optimal objective value.

(c) **[10 points]** Solve the ILP from (a) by applying the **branch-and-bound** method. Clearly indicate in which order you compute the nodes of the search tree and where you prune the search tree. Also indicate which pruning criterion you use when you prune.

Solution exercise 3:

(a)

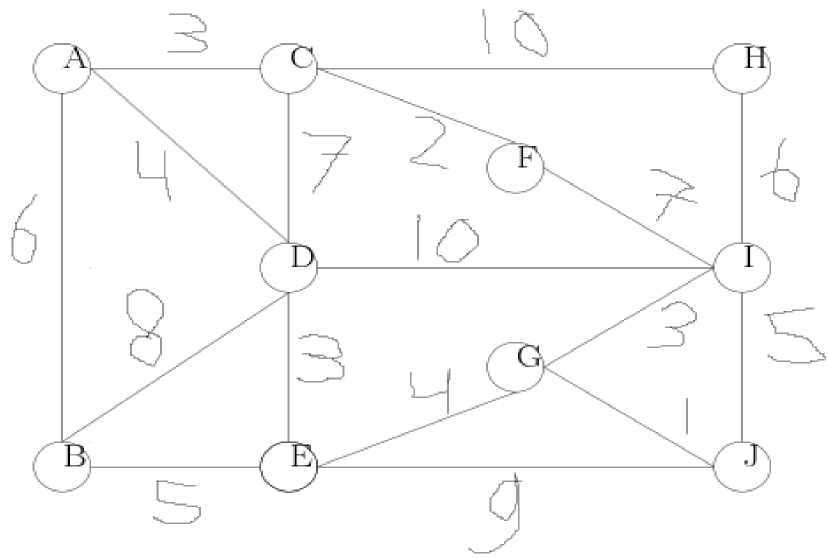
$$\begin{aligned} \max \quad & z = 10x_1 + 17x_2 + 8x_3 + 14x_4 \\ \text{s.t.} \quad & 4x_1 + 7x_2 + 3x_3 + 5x_4 \leq 10 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

(b) Since $\frac{14}{5} > \frac{8}{3} > \frac{10}{4} > \frac{17}{7}$ it follows for this binary knapsack problem that the LP relaxation has optimal solution $x_1 = \frac{1}{2}$, $x_2 = 0$, $x_3 = 1$, $x_4 = 1$ with $z^* = 27$.

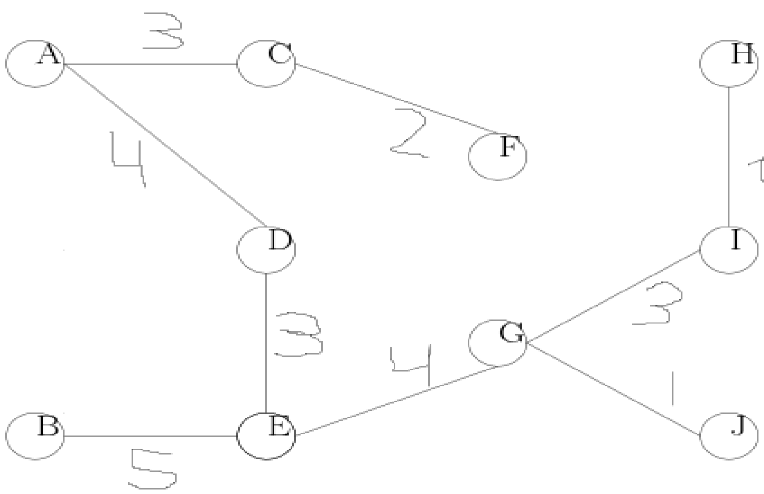
(c) From (b) we already have the upperbound $z^0 = 27$ for starting node 0. From starting node 0 we branch on variable x_1.
Put $x_1 = 0$ for subproblem 1. Then the LP relaxation has optimal solution $x_1 = 0, x_2 = \frac{2}{7}, x_3 = 1, x_4 = 1$ with $z^1 = 26\frac{6}{7}$. Branch on variable x_2 from node 1. Putting $x_1 = 0$, $x_2 = 0$ for subproblem 1.1 the LP relaxation has optimal solution $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 1$ with $z^{1.1} = 22$. This gives also an lower bound for the original problem since this LP relaxation solution is feasible.
Put $x_1 = 0$, $x_2 = 1$ for subproblem 1.2. Then the LP relaxation has optimal solution $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = \frac{3}{5}$ yielding upper bound $z^{1.2} = 25\frac{1}{5}$. Branch on variable x_4 from node 1.2.
Putting $x_1 = 0, x_2 = 1, x_4 = 0$ for node 1.2.1 yields the solution $x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0$ with value $z^{1.2.1} = 25$. This solution gives a new best lower bound.
Putting $x_1 = 0$, $x_2 = 1$, $x_4 = 1$ for node 1.2.2 gives no feasible solutions (already too much weight is used) and thus we can prune this branch. It remains to consider subproblem 2 by putting $x_1 = 1$. Then the LP relaxation has optimal solution $x_1 = 1, x_2 = 0, x_3 = \frac{1}{3}, x_4 = 1$ with $z^2 = 26\frac{2}{3}$. Branch on x_3 from node 2. Putting $x_1 = 1$, $x_3 = 1$ gives only $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$ with value $z = 18$ as optimal solution which is worse than the lowerbound. We can prune.
It now only remains to consider node 2.1 by putting $x_1 = 1$, $x_3 = 0$. Then the LP relaxation has optimal solution $x_1 = 1$, $x_2 = \frac{1}{7}$, $x_3 = 0$, $x_4 = 1$ yielding upperbound $z^{2.1} = 26\frac{3}{7}$. Branch on x_2 from node 2.1. Putting $x_2 = 1$ gives no feasible solutions (too much weight used) . It only remains to consider node 2.1.1 by putting $x_1 = 1$, $x_2 = 0$, $x_3 = 0$. Then the LP relaxation gives $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1$ with value $z^{2.1.1} = 24$. This is feasible for the original problem, but the value of 24 is worse than the lower bound 25 we have obtained before.
The conclusion is that $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$ with value $z^* = 25$ is the optimal solution of the binary knapsack problem.

Exercise 4

(a) **[10 points]** Consider the problem of finding a minimum weight spanning tree in the non-directed graph below (where edge weights have been indicated) using Prim's algorithm choosing node A as initial tree. Make clear in which order the edges are picked by the algorithm and draw the minimum weight spanning tree which is finally obtained. Explain your answer briefly.

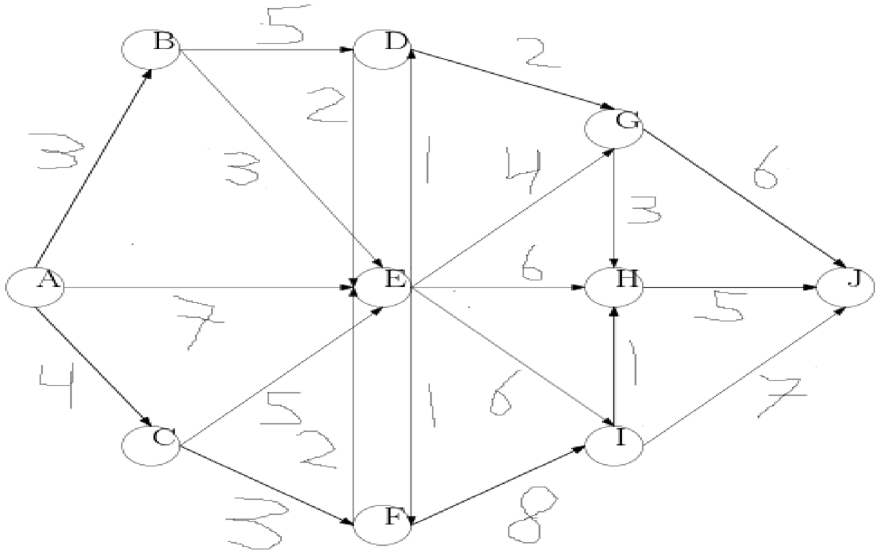


Solutions exercise 4.
Prim's algorithm adds an edge of lowest weight under the condition that the edge should be connected to the existing tree and there is no cycle formed by adding that edge. Applying this algorithm the following edges are added in the following order:
$\{A, C\}, \{C, F\}, \{A, D\}, \{D, E\}, \{E, G\}, \{G, J\}, \{G, I\}, \{B, E\}, \{I, H\}$.
The resulting minimal spanning tree is then as follows:



Exercise 5

Consider the directed graph shown below with length of the arcs as indicated.



(a) **[10 points]**. Apply Dijkstra's algorithm to determine a **shortest path** from node A to node J in this directed graph. It should be clear from your work that you have applied Dijkstra's algorithm to determine a shortest path. At the end clearly draw the shortest path which you have found.

(b) **[5 points]**. The arcs (E, D) and (E, F) are removed from the directed graph and can no longer be used on any path between two nodes. In this modified directed graph determine a **shortest path** from node A to node J by applying backward recursion. It should be clear from your work that you have applied backward recursion to determine a shortest path. At the end clearly draw the shortest path which you have found.

Solution exercise 5:

(a) The calculations from Dijkstra's algorithm are summarized in the following table where label (x, y) for a node means that x is the shortest distance from A to the node obtained until the current iteration and y is the predecessor of the node on that shortest path to the node. Once the label of a node has become permanent it is no longer written in this table in the following iterations (since it will not change anymore). If the label is followed by * it means that this label becomes permanent in the current iteration.

iteration	B	C	D	E	F	G	H	I	J
1	$(3, A)^*$	$(4, A)$		$(7, A)$					
2		$(4, A)^*$	$(8, B)$	$(6, B)$					
3			$(8, B)$	$(6, B)^*$	$(7, C)$				
4			$(7, E)^*$		$(7, C)$	$(10, E)$	$(12, E)$	$(12, E)$	
5					$(7, C)^*$	$(9, D)$	$(12, E)$	$(12, E)$	
6						$(9, D)^*$	$(12, E)$	$(12, E)$	
7							$(12, E)^*$	$(12, E)$	$(15, G)$
8								$(12, E)^*$	$(15, G)$
9									$(15, G)^*$

Backtracking from node J we obtain that the corresponding shortest path is: $A \to B \to E \to D \to G \to J$ having total length 15.

(b) After removing the arcs (E, D) and (E, F) the directed graph is acyclic and then it is possible to number the nodes such that for every arc (i, j) in the graph it holds that $i < j$. Such numbering is for example $A = 1,\ B = 2,\ C = 3,\ D = 4,\ F = 5,\ E = 6,\ G = 7,\ I = 8,\ H = 9,\ J = 10$.

Now define the value function $f(i)$ to be the length of the shortest path from node numbered i to destination node $J = 10$. Initialize $f(10) = 0$ and compute the other function values by the backward recursion $f(i) = \min_{j:(i,j)\in A}[w(i, j) + f(j)]$.
Then it follows consecutively (doing calculations in reverse order of the numbering of the nodes) that $f(9) = 5,\ f(8) = 6,\ f(7) = 6,\ f(6) = 10\ ,\ f(5) = 12,\ f(4) = 8,\ f(3) = 15,\ f(2) = 13,\ f(1) = 16$.
Backtracking we obtain as shortest path either the path $A \to B \to D \to G \to J$ or $A \to B \to E \to G \to J$. It is easily seen that the length of these paths are indeed 16 corresponding with the calculated function value $f(1)$.