**Exam Operations Research**

Date: June11, 2020
Time: 10:00 - 12:15

**Points per exercise**:

- Exercise 1 has a total of 25 points (a(10), b(5) and c(10)).
- Exercise 2 has a total of 15 points.
- Exercise 3 has a total of 20 points (a(5), b(5) and c(10)).
- Exercise 4 has a total of 20 points (a(10) and b(10)).
- Exercise 5 has a total of 10 points.

Thus in total 90 points can be obtained. The exam grade is determined as follows:

$$\text{Exam grade } = 1 + \frac{\text{total number of obtained points}}{10}.$$

- Calculator is allowed.
- This exam consists of 5 pages, including this one.
- The duration of this exam is **2 hours and 15 minutes**. When you finish working you notify the host of your meeting. Within 15 minutes after you finish working you have to upload the scanned pdf file in Canvas in the assignment for the June 11 exam. After uploading the pdf file you notify the host again.
- Students who have obtained permission for extra time may use an *additional 30 minutes*.

**Exercise 1**

Consider the following LP which is referred to as the "primal LP".

$$\begin{aligned} \max \quad & z = x_1 + 3x_2 - x_3 + x_4 \\ \text{s.t.} \quad & x_1 + 2x_2 - 2x_3 + x_4 \leq 10 \\ & x_1 + 2x_2 + x_3 = 20 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(a) **[10 points]** Determine the dual of this LP.

(b) **[5 points]** The primal LP has been solved by the simplex method resulting in the following final tableau:

| Basic | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $r_1$ | value |
|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 30 |
| $x_4$ | 0 | 3 | 6 | 0 | 1 | 1 | 2 | 50 |
| $x_3$ | 0 | 1 | 2 | 1 | 0 | 0 | 1 | 20 |

Determine the optimal solution of the primal LP and optimal objective value. Determine also the optimal solution of the dual LP and optimal dual objective function.

(c) **[10 points]** The extra constraint $x_4 \leq 20$ is added to the primal LP. Apply the dual simplex method to determine the optimal solution and optimal objective value of the modified LP.
*Instruction: If you do the calculations correctly you get some fractions in the simplex tableau, but all these fractions are of the form* $\frac{1}{6}k$ *with* $k$ *integer. Moreover, it will require only one pivot step to find the new optimal solution.*

**Solution exercise 1**:

(a) The dual LP is:

$$
\begin{aligned}
\min \quad & w = 10y_1 + 20y_2 \\
\text{s.t.} \quad & y_1 + y_2 \geq 1 \\
& 2y_1 + 2y_2 \geq 3 \\
& -2y_1 + y_2 \geq -1 \\
& y_1 \geq 1 \\
& y_1 \geq 0, y_2 \text{ unrestricted}
\end{aligned}
$$

(b) The optimal solution of the primal LP is $x_1^* = 0$, $x_2^* = 0$, $x_3^* = 20$, $x_4^* = 50$ and optimal objective value $z^* = 30$. The optimal solution of the dual LP is $y_1^* = 1$, $y_2^* = 1$ with optimal dual objective value $w^* = 30$.

(c) The modified simplex tableau is:

| Basic | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $r_1$ | $s_2$ | value |
|---|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 30 |
| $x_4$ | 0 | 3 | 6 | 0 | 1 | 1 | 2 | 0 | 50 |
| $x_3$ | 0 | 1 | 2 | 1 | 0 | 0 | 1 | 0 | 20 |
| $s_2$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 20 |

Subtracting the $x_4$ row from the $s_2$ row to get the appropriate unit vector in the $x_4$ column we obtain:

| Basic | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $r_1$ | $s_2$ | value |
|---|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 30 |
| $x_4$ | 0 | 3 | 6 | 0 | 1 | 1 | 2 | 0 | 50 |
| $x_3$ | 0 | 1 | 2 | 1 | 0 | 0 | 1 | 0 | 20 |
| $s_2$ | 0 | $-3$ | $-6$ | 0 | 0 | $-1$ | $-2$ | 1 | $-30$ |

Now we apply one iteration by the dual simplex method. Since $s_2$ is negative in current solution it is leaving the basis. $x_2$ has the minimal ratio among the variables with negative coefficient in $s_2$ row and therefore enters the basis. Then we obtain the following tableau:

| Basic | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $r_1$ | $s_2$ | value |
|---|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | $\frac{1}{2}$ | 0 | 0 | 0 | $\frac{5}{6}$ | $\frac{2}{3}$ | $\frac{1}{6}$ | 25 |
| $x_4$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 20 |
| $x_3$ | 0 | 0 | 0 | 1 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 10 |
| $x_2$ | 0 | $\frac{1}{2}$ | 1 | 0 | 0 | $\frac{1}{6}$ | $\frac{1}{3}$ | $-\frac{1}{6}$ | 5 |

From this tableau it follows that the optimal solution of the modified LP is $x_1^* = 0$, $x_2^* = 5$, $x_3^* = 10$, $x_4^* = 20$ with optimal objective value $z^* = 25$.

**Exercise 2**

There are 3 distribution centers ($i = 1, 2, 3$) from which a certain product can be transported to fulfill the weekly demand for this product of 4 towns ($j = 1, 2, 3, 4$). Town $j$ has a weekly demand for this product of $D_j$ tons. The cost of transporting the product from distribution center $i$ to town $j$ is $C_{ij}$ per ton. Moreover, distribution centre $i$ has a startup cost $A_i$ (per week) if this product is transported from distribution centre $i$ to at least one town. In other words the weekly startup cost $A_i$ is not incurred if there is no transport of this product from distribution centre $i$. For each distribution centre $i$ it is not possible to transport more than $B_i$ tons of the product per week.

(a) **[15 points]** Formulate the problem of minimizing the total (of transport and startup) costs per week such that the demand for the product is fulfilled in each town as a mixed integer linear program. Explain the meaning of all variables and constraints you have in your mixed integer linear program formulation.

**Solution exercise 2**:
Let $x_{ij}$ be the amount of tons transported per week from distribution center $i$ to town $j$. Let binary variable $y_i$ indicate whether there is any transportation from distribution center $i$. Let $M$ be some number which is large enough ($M$ is large enough if it is greater or equal than all $B_i$). Then we can formulate the problem as follows:

$$\min \quad w = \sum_{i=1}^{3}\sum_{j=1}^{4} C_{ij}x_{ij} + \sum_{i=1}^{3} A_i y_i$$

$$\text{s.t.} \quad \sum_{i=1}^{3} x_{ij} = D_j \qquad \text{for } j = 1, 2, 3, 4$$

$$\sum_{j=1}^{4} x_{ij} \leq B_i \qquad \text{for } i = 1, 2, 3$$

$$\sum_{j=1}^{4} x_{ij} \leq M y_i \qquad \text{for } i = 1, 2, 3$$

All variables $x_{ij} \geq 0$, all variables $y_i \in \{0, 1\}$

**Exercise 3**

Consider the following problem. You are packing your suitcase for a journey and need to decide what items to bring. You have 4 items (just one of each); each has a certain value to you, and a certain weight:

| | | | | |
|---|---|---|---|---|
| weight (kg) | 4 | 7 | 8 | 11 |
| value | 8 | 15 | 18 | 24 |

You do not want to have more than 16 kg of weight in the suitcase. The problem is to choose under that weight restriction which items should be put into the suitcase maximizing the total value of those items.

(a) [**5 points**] Formulate an integer linear program for this problem.

(b) [**5 points**] Determine the unique optimal solution of the LP relaxation of this problem. Also determine the corresponding optimal objective value.

(c) [**10 points**] Solve the ILP from (a) by applying the **branch-and-bound** method. Clearly indicate in which order you compute the nodes of the search tree and where you prune the search tree. Also indicate which pruning criterion you use when you prune.

**Solution exercise 3**:

(a)

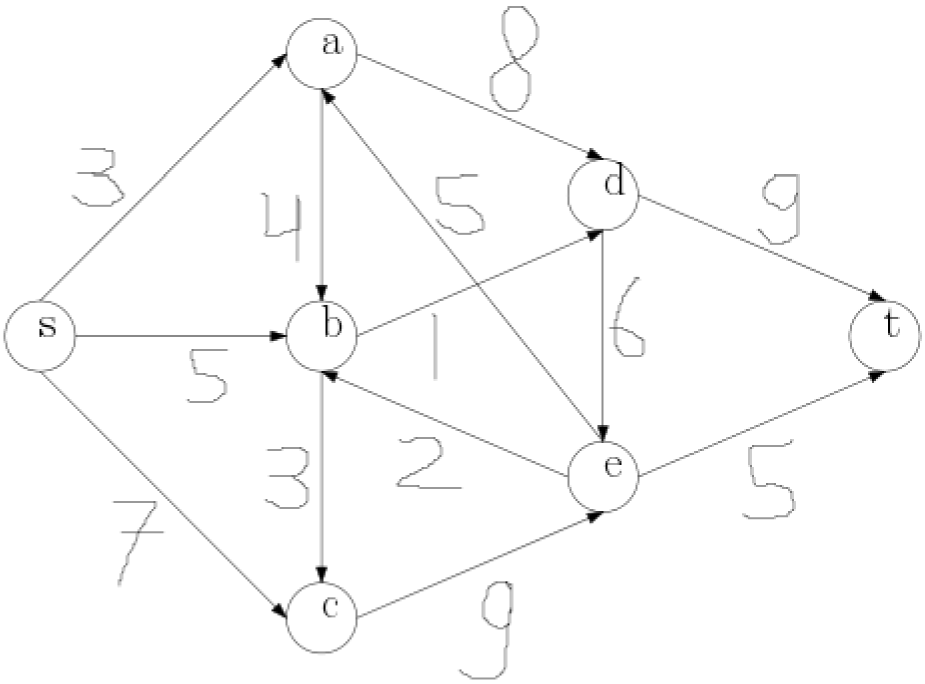
$$\begin{array}{lll} \max & z = 8x_1 + 15x_2 + 18x_3 + 24x_4 & \\ & \text{s.t.} \quad 4x_1 + 7x_2 + 8x_3 + 11x_4 \leq 16 & \\ & \qquad\qquad x_1, x_2, x_3, x_4 \in \{0, 1\} & \end{array}$$

(b) Since $\frac{18}{8} > \frac{24}{11} > \frac{15}{7} > \frac{8}{4}$ it follows for this binary knapsack problem that the LP relaxation has optimal solution $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = \frac{8}{11}$ with $z^* = 35\frac{5}{11}$.

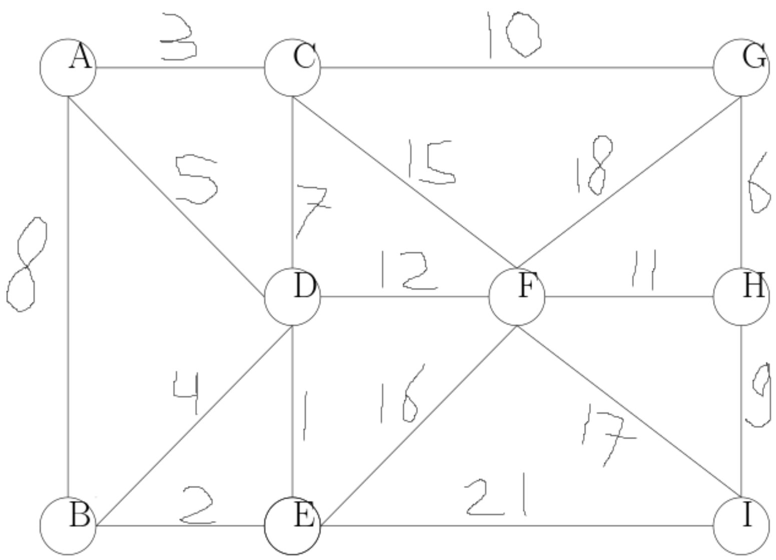
(c) From (b) we already have an upperbound $z^0 = 35$ since the value is always integer for feasible solutions. From starting node 0 we branch on variable $x_4$.
Put $x_4 = 0$ for subproblem 1. Then the LP relaxation has optimal solution $x_1 = \frac{1}{4}, x_2 = 1, x_3 = 1, x_4 = 0$ yielding upper bound $z^1 = 35$. Branch on variable $x_1$ from node 1. Put $x_1 = 0$, $x_4 = 0$ for subproblem 1.1. Then the LP relaxation has optimal solution $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$ yielding upper bound $z^{1.1} = 33$. This is also an lower bound for the original problem since this LP relaxation solution is feasible.
Put $x_1 = 1$, $x_4 = 0$ for subproblem 1.2. Then the LP relaxation has optimal solution $x_1 = 1$, $x_2 = \frac{4}{7}$, $x_3 = 1$, $x_4 = 0$ yielding upper bound $z^{1.2} = 34\frac{4}{7}$. Branch on variable $x_2$ from node 1.2.
Putting $x_1 = 1, x_2 = 0, x_4 = 0$ for node 1.2.1 yields the solution $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$ with value $z^{1.2.1} = 26$. This solution is worse than the earlier obtained lowerbound 33 from node 1.1.
Putting $x_1 = 1$, $x_2 = 1$, $x_4 = 0$ for node 1.2.2 gives $x_1 = 1$, $x_2 = 1$, $x_3 = 0$ , $x_4 = 0$ with value 23 as remaining feasible solution.
Therefore prune and it remains only to consider subproblem 2 by putting $x_4 = 1$. Then the LP relaxation has optimal solution $x_1 = 0, x_2 = 0, x_3 = \frac{5}{8}, x_4 = 1$ with $z^2 = 35\frac{1}{4}$. Branch on $x_3$ from node 2. Notice that putting $x_3 = 1$, $x_4 = 1$ gives no feasible solutions (already too much weight is used) and thus prune that branch.
It now only remains to consider node 2.1 by putting $x_3 = 0$, $x_4 = 1$. Then the LP relaxation has optimal solution $x_1 = 0$, $x_2 = \frac{5}{7}$, $x_3 = 0$ , $x_4 = 1$ yielding upperbound $z^{2.1} = 34\frac{5}{7}$. Branch on $x_2$ from node 2.1. Putting $x_2 = 1$ gives no feasible solutions thus consider node 2.1.1 by putting $x_2 = 0$, $x_3 = 0$, $x_4 = 1$. Then the LP relaxation gives $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1$ as optimal solution. This is feasible for the original problem, but the corresponding value is 32 is worse than the lower bound 33 we have obtained before.
The conclusion is that $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$ with value $z^* = 33$ is the optimal solution of the binary knapsack problem.

**Exercise 4**

(a) **[10 points]** Consider the instance of the maximum flow problem shown in the directed graph below where the arc capacities are indicated).
Let the current flow $f$ (which is feasible but not maximal) be as follows: $f_{sa} = 3$, $f_{sb} = 4$, $f_{sc} = 5$, $f_{ad} = 6$, $f_{bd} = 1$, $f_{bc} = 3$, $f_{ce} = 8$, $f_{dt} = 7$, $f_{ea} = 3$, $f_{et} = 5$ and no flow on all other arcs of the given graph. Draw the residual graph $D^f$ corresponding to this flow $f$. Continue from this residual graph the Ford-Fulkerson algorithm to determine a maximum flow from $s$ to $t$. State the value of the flow and show that it is maximal by giving a minimum $s$-$t$ cut of that value.

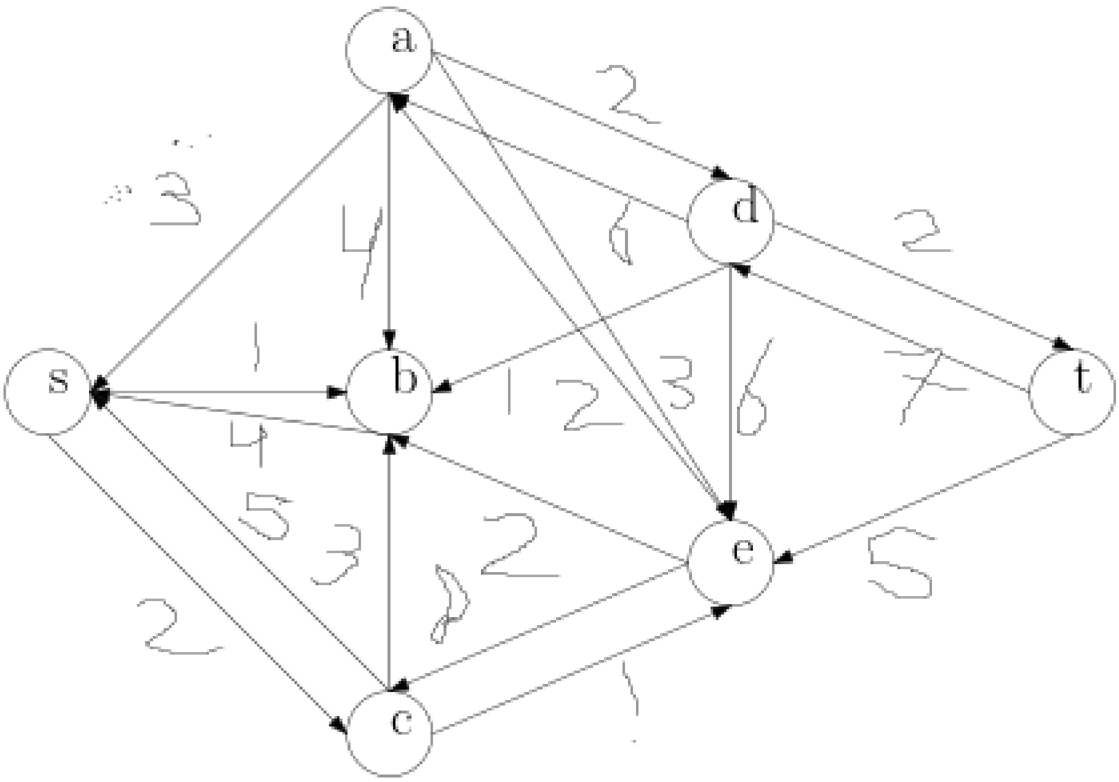



(b) **[10 points]** Consider the problem of finding a minimum weight spanning tree in the non-directed graph below (where edge weights have been indicated) using Kruskal's algorithm. Make clear in which order the edges are picked by the algorithm and draw the minimum weight spanning tree which is finally obtained. Explain your answer briefly.



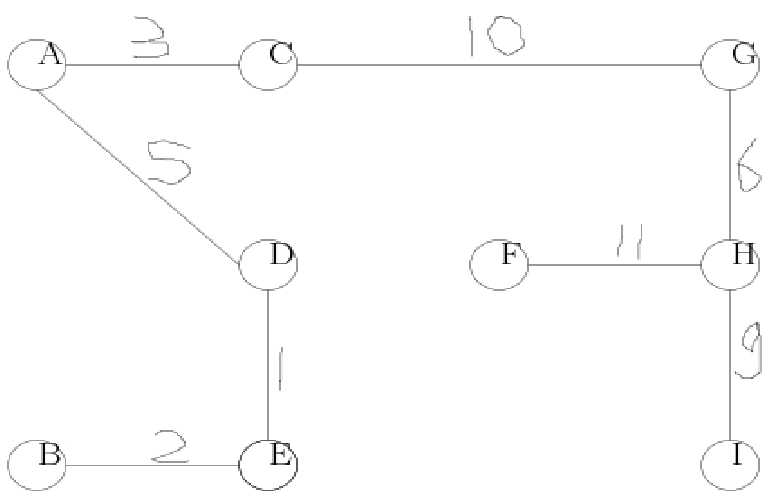
**Solutions exercise 4.**

(a) The residual graph for the given flow is as follows:




An augmenting path in the above residual graph is: $s \to c \to e \to a \to d \to t$ on which an extra flow of maximal 1 can be pushed. Pushing this extra flow of 1 will remove the arc from $c$ to $e$ in the next residual graph. It is easily seen that in that next residual graph there will be no longer a path from $s$ to $t$ because only nodes $b$ and $c$ can be reached from $s$. Thus the resulting flow after pushing 1 on the augmenting path should be maximal. This resulting flow is: $f_{sa} = 3$, $f_{sb} = 4$, $f_{sc} = 6$, $f_{ad} = 7$, $f_{bd} = 1$, $f_{bc} = 3$, $f_{ce} = 9$, $f_{dt} = 8$, $f_{ea} = 4$, $f_{et} = 5$ having value 13. To show that this is indeed a maximal flow an $s - t$ cut in the original graph of the same total capacity of 13 should be provided. This is the cut consisting of the arcs $(s, a), (b, d), (c, e)$ which indeed have a total capacity of $3 + 1 + 9 = 13$ and which is a cut between the nodes $\{s, b, c\}$ and the other nodes.

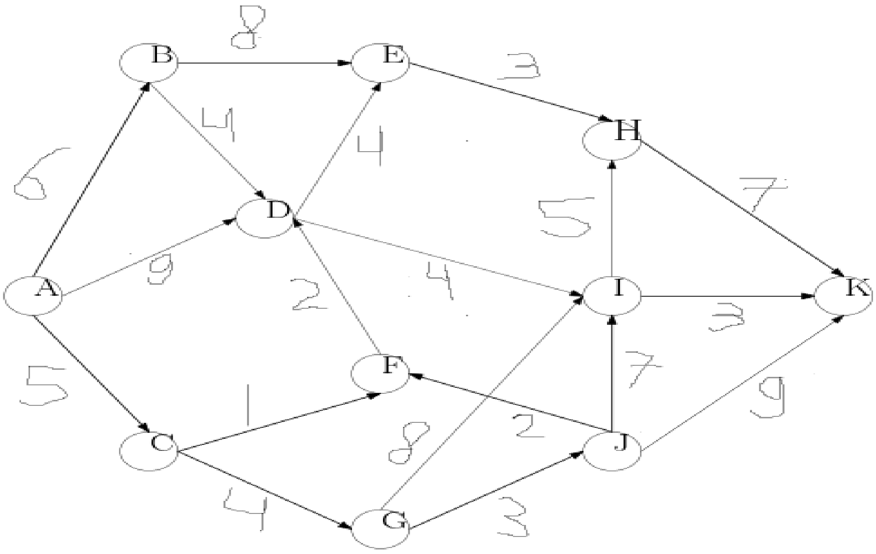
(b) Kruskal's algorithm adds the edge with the lowest weight to the tree unless a cycle is formed by adding that edge. Applying this algorithm the following edges are added in the following order: $\{D, E\}, \{B, E\}, \{A, C\}, \{A, D\}, \{G, H\}, \{H, I\}, \{C, G\}, \{F, H\}$. The resulting minimal spanning tree is then as follows:



**Exercise 5**

Consider the acyclic directed graph shown below with length of the arcs as indicated.

(a) **[10 points]**. Apply dynamic programming to determine the **longest path** from node $A$ to node $K$ in this directed graph. It should be clear that you have applied dynamic programming. At the end present clearly the longest path which you have found.




**Solution exercise 5**:
Since not all paths from $A$ to $K$ have the same number of arcs it makes no sense to define stages and apply recursion using the stages. Also Dijkstra's algorithm should not be used to determine the longest path. Instead because the directed graph is acyclic we can number the 11 nodes in the graph such that there are only forward arcs with respect to that numbering. After such numbering of the nodes backward recursion can be applied using the numbering of the nodes. Such a numbering which is applicable for backward recursion is $A = 1$, $B = 2$, $C = 3$, $G = 4$, $J = 5$, $F = 6$, $D = 7$, $E = 8$, $I = 9$, $H = 10$, $K = 11$.
Define the value function $f(i)$ to be the length of the longest path from node numbered $i$ to destination node $K = 11$. Initialize $f(11) = 0$ and compute the other function values in backward order by the recursion $f(i) = \max_{j:(i,j)\in A}[w(i,j) + f(j)]$. Then it follows consecutively (doing calculations in reverse order of the numbering of the nodes) that $f(10) = 7$, $f(9) = 12$, $f(8) = 10$, $f(7) = 16$ , $f(6) = 18$, $f(5) = 20$, $f(4) = 23$, $f(3) = 27$, $f(2) = 20$, $f(1) = 32$.
Backtracking we obtain as longest path the path $A \to C \to G \to J \to F \to D \to I \to H \to K$. It is easily checked that the length of this path is indeed 32 corresponding with the function value $f(1)$.