Exam Operations Research

Date: June 4, 2019
Time: 15:30 - 18:15

Points per exercise:

- Exercise 1 has a total of 25 points (a(10), b(5) and c(10)).
- Exercise 2 has a total of 15 points.
- Exercise 3 has a total of 20 points (a(5), b(5) and c(10)).
- Exercise 4 has a total of 15 points (a(5), b(5) and c(5)).
- Exercise 5 has a total of 15 points (a(5) and b(10)).

Thus in total 90 points can be obtained. The exam grade is determined as follows:

$$\text{Exam grade } = 1 + \frac{\text{total number of obtained points}}{10}.$$

$$\text{Final course grade} = \frac{1}{4}\text{Weekly pretest grade} + \frac{3}{4}\text{Exam grade.}$$

- Non-programmable, non-graphing calculators are allowed.
- This exam consists of 4 pages, including this one.
- The duration of this exam is **2 hours and 45 minutes**.

  Students who have obtained permission for extra time may use an *additional 30 minutes*. This is only possible in the room allocated for this purpose.

Exercise 1

Consider the following LP which is referred to as the "primal LP".

$$\begin{aligned} \max \quad & z = x_1 + 2x_2 - 2x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 - x_3 - x_4 \leq 10 \\ & 3x_1 + 2x_2 + 2x_4 \leq 24 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(a) **[10 points]** Translate this LP-problem into standard basis form. Next give the starting solution for the simplex method and write down the starting tableau. Then apply one iteration with x_2 as basis entering variable to obtain an improved basic feasible solution. Explain carefully which variable leaves the basis while performing this iteration. What is the new basic feasible solution you have obtained?

(b) **[5 points]** The primal LP has been solved by the simplex method resulting in the following final tableau:

Basic	z	x_1	x_2	x_3	x_4	s_1	s_2	value
z	1	2.50	0	1.00	0	1.00	0.50	22.00
x_2	0	1.75	1.00	-0.50	0	0.50	0.25	11.00
x_4	0	-0.25	0	0.50	1.00	-0.50	0.25	1.00

Explain how from this tableau it is clear that the corresponding solution is optimal. Give the optimal solution of the primal LP and optimal objective value.

(c) **[10 points]** Determine the dual LP of the given primal LP. Next give the optimal solution of this dual LP and optimal dual objective value.

Exercise 2 [15 points]

Three adventurers have found a treasure containing ten valuable items. They estimate the value of item number i, $i = 1, 2, \ldots, 10$, to be some positive integer v_i. Now they want to divide these ten items among themselves. They agree on the following procedure to divide the ten items.

The ten items will be divided among three boxes j, $j = 1, 2, 3$, in a fair way. After that each of them gets by drawing of lots the content of one of the boxes. To divide the items in a fair way among the three boxes they impose the following conditions. First of all each box should get at least two and at most four items. Moreover, since the items numbered 1,2 and 3 are the most interesting for these adventurers they impose that each box should get exactly one of these three items. Under these conditions they want that the total value of the items in each box should be as fair as possible. Therefore they want under above conditions that the difference between the total value of the items in the box that gets the most total value and the box that gets the least total value is minimized.

Formulate the problem of minimizing the difference in value between the box getting the most total value and the box getting the least total value satisfying the imposed conditions as an integer linear program. Explain the meaning of all variables and constraints you have in your integer linear program formulation.

Instruction: By free choice in numbering the three boxes you are allowed to impose restrictions which make sure that the total value in box number1 will be at least as much as the total value in box number 2 and the total value in box number 2 will be at least as much as the total value in box number 3. Imposing these restrictions makes it more straightforward to formulate the integer linear program.

Exercise 3

Consider the following problem. You are packing your suitcase for a journey and need to decide what items to bring. You have 4 items (just one of each); each has a certain value to you, and a certain weight:

weight (kg)	7	10	15	21
value	10	14	23	32

You do not want to have more than 32 kg of weight in the suitcase. The problem is to choose under that weight restriction which items should be put into the suitcase maximizing the total value of those items.

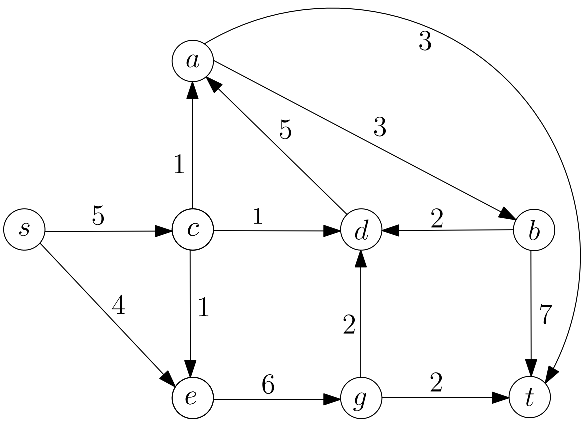
(a) **[5 points]** Formulate an integer linear program for this problem.

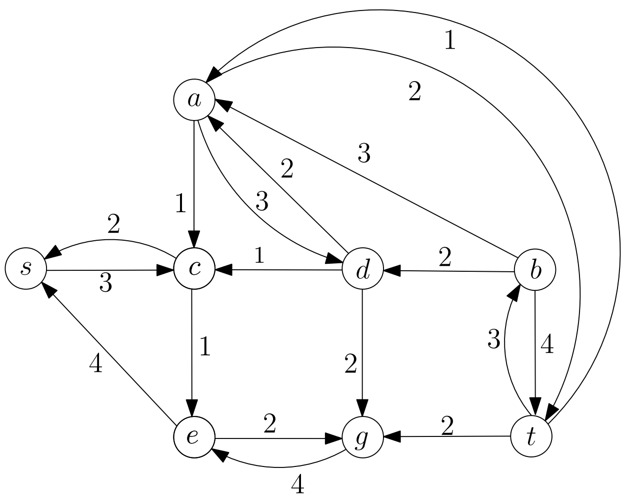
(b) **[5 points]** Determine the unique optimal solution of the LP relaxation of this problem. Also determine the corresponding optimal objective value.

(c) **[10 points]** Solve the ILP from (a) by applying the **branch-and-bound** method. Clearly indicate in which order you compute the nodes of the search tree and where you prune the search tree. Also indicate which pruning criterion you use when you prune.

Exercise 4

Consider the directed graph $D = (V, A)$ with capacities u, shown on the left. On the right, the **residual graph** D^f (with capacities) corresponding to some s-t-flow f is shown. (The actual flow f is **not** shown.)

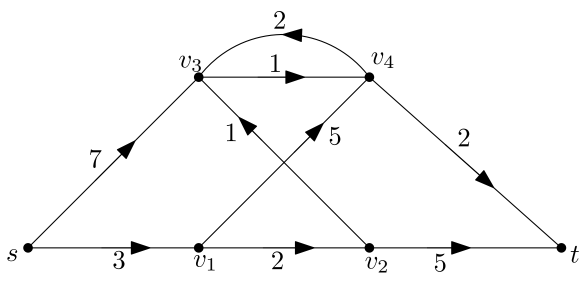




(a) **[5 points]** State the value of the flow f, as well as the values of f_{da}, f_{bd} and f_{at}.

(b) **[5 points]** Either give a convincing argument that f is a maximum flow, or explain how f can be adjusted to give a flow f' of strictly larger value.

(c) **[5 points]** *Using Dijkstra's algorithm*, find a shortest path from s to t in the following directed graph, where the arc lengths have been indicated.



(To obtain credit, it must be clear from your working that you are using Dijkstra's algorithm.)

Exercise 5

(a) **[5 points]** *Using dynamic programming*, determine a longest increasing subsequence of

$$4, 6, 7, 1, 3, 2, 8, 9, 5.$$

You should find both the length of the longest increasing subsequence, as well as the subsequence itself.

(To obtain credit, it must be clear from your working that you are using dynamic programming.)

(b) **[10 points]** Now, suppose we want to find not a *longest* increasing subsequence, but an increasing subsequence of largest *sum*. For example: in the above sequence, $(1, 2, 5)$ and $(3, 8, 9)$ are both increasing subsequences of length 3, but the first has sum $1 + 2 + 5 = 8$ and the second has sum $3 + 8 + 9 = 20$.

Give a dynamic programming algorithm for the problem of finding an increasing subsequence of largest possible sum, by filling in the missing parts ([[1]], [[2]]) of the template below. You must also describe what the meaning of the value $K(i)$ is in your solution.

Input: sequence $(x_1, x_2, \ldots, x_n)$.
for $i = 1, 2, \ldots, n$ **do**
    $K(i) =$ [[1]]
return [[2]]