Operations Research Resit

2h45m

- This exam consists of 6 exercises, from which 100 points can be obtained in total. The division of the points over the various parts is as follows:

Question:	1	2	3	4	5	6	Total
Points:	30	20	10	10	15	15	100
Score:							

- **Exam grade**: $\frac{\text{total number of points}}{10}$

  Final course grade: $\frac{1}{4}$Weekly pretest grade + $\frac{3}{4}$Exam grade

- Non-programmable, non-graphing calculators are allowed.

- This exam consists of 8 pages, including this one.

- The duration of this exam is **2 hours and 45 minutes**.

  Students who have obtained permission for extra time may use an *additional 30 minutes*. This is only possible in the room allocated for this purpose.

Good luck!

1. Consider the following LP, which we will refer to as the "primal LP".

$$
\begin{aligned}
\max \quad & t \\
\text{s.t.} \quad & t \leq 2x_1 - x_2 \\
& t \leq -x_1 - 2x_2 + 3x_3 \\
x_1 + x_2 + x_3 &= 1 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

(a) (10 points) Determine the dual of this LP.

Solution: First move all variables to the LHS:

$$
\begin{aligned}
\max \quad & z \\
\text{s.t.} \quad t - 2x_1 + x_2 &\leq 0 \\
t + x_1 + 2x_2 - 3x_3 &\leq 0 \\
x_1 + x_2 + x_3 &= 1 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

The dual is

$$
\begin{aligned}
\min \quad & w \\
\text{s.t.} - 2y_1 + y_2 + w &\geq 0 \\
y_1 + 2y_2 + w &\geq 0 \\
-3y_2 + w &\geq 0 \\
y_1 + y_2 &= 1 \\
y_1, y_2 &\geq 0
\end{aligned}
$$

(b) (2 points) State the theorem of strong duality.

Solution: Bookwork.

(c) (7 points) The primal LP can be interpreted as describing the best mixed strategy for a player (call her Alice) against another player (Bob).

(i) Briefly describe the game (you should write down a matrix as part of your description).

Solution: Consider the matrix

$$
A = \begin{pmatrix} 2 & -1 \\ -1 & -2 \\ 0 & 3 \end{pmatrix}
$$

> Alice must choose a row of the matrix, and Bob a column. Alice receives the resulting entry of A from Bob (or if negative, gives minus that amount to Bob). The variable x_i represents the probability that Alice chooses row i, and z her expected payoff.

(ii) What interpretation does the dual program have in this game?

> **Solution:** It represents the problem of choosing a best mixed strategy for Bob.

Now consider the following simplex tableau, which represents a basic feasible solution to the primal LP obtained after running the first phase of the two-phase simplex method:

Basic	z	t	x_1	x_2	x_3	s_1	s_2	value
(z)	1	-1	0	0	0	0	0	0
(x_1)	0	$-\frac{2}{7}$	1	0	0	$-\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$
(x_2)	0	$\frac{3}{7}$	0	1	0	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{3}{7}$
(x_3)	0	$-\frac{1}{7}$	0	0	1	$\frac{1}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$

(d) (2 points) Write down the basic feasible solution and objective value that is represented by this tableau.

> **Solution:** The objective value is 0. $x_1 = 3/14, x_2 = 3/7, x_3 = 5/14$, $s_1 = s_2 = 0$.

(e) (9 points) Perform a **single step** of the simplex method starting from the above tableau, and show the resulting tableau. Write down the solution represented by this new tableau, and explain whether or not it is an optimal solution to the LP.

> **Solution:**
>
> | Basic | z | t | x_1 | x_2 | x_3 | s_1 | s_2 | value |
> |---|---|---|---|---|---|---|---|---|
> | (z) | 1 | 0 | 0 | $\frac{7}{3}$ | 0 | $\frac{2}{3}$ | $\frac{1}{3}$ | 1 |
> | (x_1) | 0 | 0 | 1 | $\frac{2}{3}$ | 0 | $-\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{2}$ |
> | (t) | 0 | 1 | 0 | $\frac{7}{3}$ | 0 | $\frac{2}{3}$ | $\frac{1}{3}$ | 1 |
> | (x_3) | 0 | 0 | 0 | $\frac{1}{3}$ | 1 | $\frac{1}{6}$ | $-\frac{1}{16}$ | $\frac{1}{2}$ |
>
> Yes, this is optimal, since all coefficients in the objective row are nonnegative.
> The optimal solution is thus $x_1 = x_3 = \frac{1}{2}$, $x_2 = 0$, $z = t = 1$.

2. Consider the following problem. You are packing your suitcase for a flight on a budget airline, and need to decide what items to bring in your single bag. You have 4 items (just one of each); each has a certain value to you, and a certain weight:

weight (kg)	10	8	6	5
value	24	17	12	6

The airline only allows a maximum weight of 15kg. The problem is to decide which items you should put into your suitcase to maximize the overall value.

(a) (4 points) Write down an ILP for this problem.

Solution:

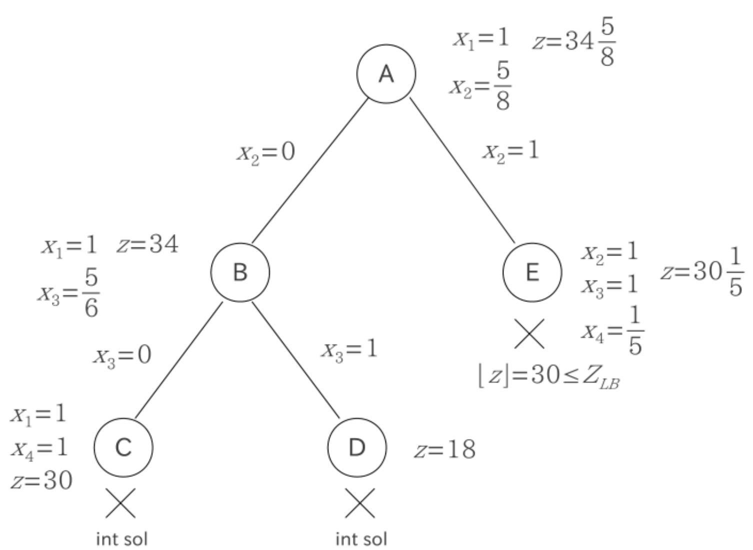
$$\begin{aligned} \max \quad & 24x_1 + 17x_2 + 12x_3 + 6x_4 \\ \text{s.t.} \quad & 10x_1 + 8x_2 + 6x_3 + 5x_4 \leq 15 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

(b) (4 points) Determine the unique optimal solution of the LP relaxation of this problem. Also give the corresponding optimal objective value.

Solution: Order the items by decreasing profit per unit weight: 1,2,3,4 (already sorted). The greedy algorithm yields then $x_1 = 1$, $x_2 = 5/8$, $x_3 = x_4 = 0$. The objective value is $24 + 17 \cdot \frac{5}{8} = 34\frac{5}{8}$.

(c) (12 points) Solve the ILP from (a) using Branch and Bound. Clearly indicate in which order you compute the nodes of the search tree and where you prune the search tree, and based on which pruning criterion. Make sure to prune as soon as possible!

(Suggestion: expand branches obtained by not selecting an item before considering the branch obtained by selecting the item.)



Solution:

Ⓐ

$$\begin{aligned} \max \quad & 24x_1 + 17x_2 + 12x_3 + 6x_4 \\ & 10x_1 + 8x_2 + 6x_3 + 5x_4 \leq 15 \\ & 0 \leq x_j \leq 1, j = 1, \cdots, 4 \end{aligned}$$

$$x_1 = 1$$
$$x_3 = \frac{5}{6}$$
$$z = 24 + \frac{12 \times 5}{6} = 34$$

Ⓒ

$$\begin{array}{lrcrcl} \max & 24x_1 & + & 6x_4 & & \\ & 10x_1 & + & 5x_4 & \leq & 15 \end{array}$$

$$x_1 = x_4 = 1$$
$$z = 30$$

Ⓓ

$$\begin{array}{lrcrcrc} \max & \cancel{24x_1} & + & 6x_4 & + & 12 \\ & \cancel{10x_1} & + & 5x_4 & \leq & 9 \\ & & & x_4 & = & 1 \end{array}$$

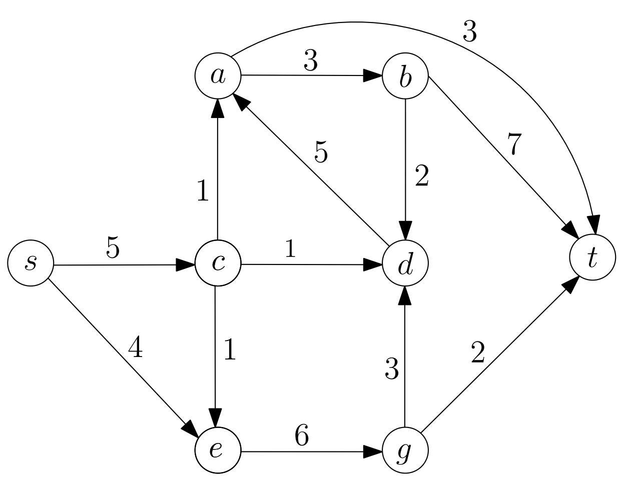
$$x_3 = x_4 = 1$$
$$z = 18$$

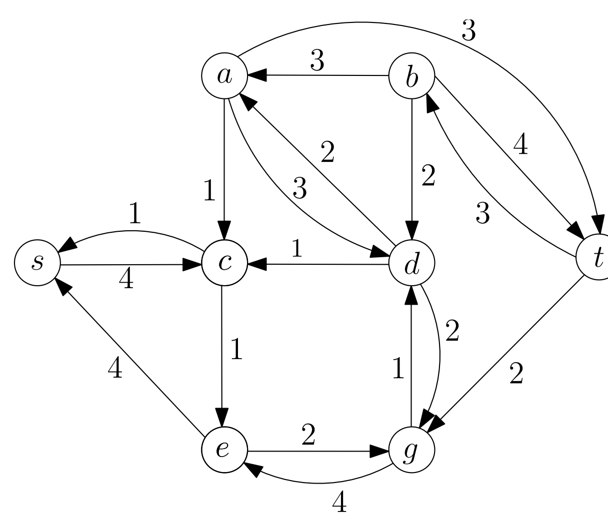
Ⓔ

$$\begin{array}{lrcrcrcr} \max & \cancel{24x_1} & + & 12x_3 & + & 6x_4 & + & 17 \\ & \cancel{10x_1} & + & 6x_3 & + & 5x_4 & \leq & 7 \end{array}$$

$$x_3 = 1$$
$$x_4 = \frac{1}{5}$$
$$z = 17 + 12 + \frac{6}{5}$$

3. Consider the directed graph $D = (V, A)$ with capacities u, shown on the left. On the right, the **residual graph** D^f (with capacities) corresponding to some s-t-flow f is shown. (The actual flow f is **not** shown.)





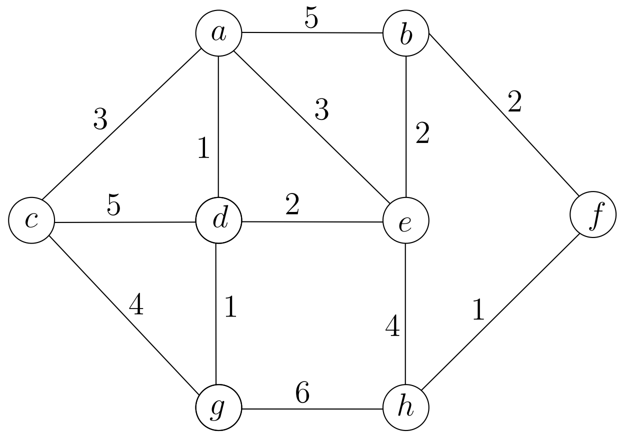
(a) (4 points) State the value of the flow f, as well as the values of f_{sc} and f_{da}.

Solution: Flow value is 5. $f_{sc} = 1$ and $f_{da} = 3$.

(b) (6 points) Either give a convincing argument that f is a maximum flow, or explain how f can be adjusted to give a flow f' of strictly larger value.

Solution: It is not a maximum flow; there is a directed s-t-path in the residual graph, namely s-c-e-g-d-a-t. Augmenting along this path yields a flow of larger value.

4. (10 points) Find a minimum cost spanning tree in the following graph, with edge costs shown. You *must* use Kruskal's algorithm, and you must show your work.



Solution: TODO

5. Recall the notion of the *edit distance* between two words: an edit operation consists of adding, removing or changing a single letter, and the edit distance is the fewest number of edit operations needed to transform the first word to the second.

   (a) (10 points) Determine the edit distance between STALL and TABLE. You should determine both the distance, and precisely *how* the first word can be transformed into the second word with this number of edits (inserting a letter, deleting a letter, or changing a letter). You *must* use dynamic programming, and you must show your working.

   Solution:

		T	A	B	L	E
	0	1	2	3	4	5
S	1	1	1			
T	2					
A	3					
L	4					
L	5					

   (b) (5 points) Suppose you had a program sitting on your computer that can find the shortest path between two given nodes in a directed graph. Explain, in a few sentences (images may also help) how you can use this program to help you solve the edit distance problem between any two given words.

   (This is a conceptual question – you do not need to do any calculations. You need to explain the relationship between the edit distance problem and the shortest path problem.)

Solution: Construct a directed graph as follows: we have a node (i, j) for each $0 \le i \le m$, $0 \le j \le n$ (where m is the length of the first word, n the length of the second word). We include an arc from (i, j) to $(i+1, j)$, $(i, j+1)$ and $(i+1, j+1)$ (whenever both endpoints exist). Each arc gets a cost of 1, except that an arc (i, j) to $(i+1, j+1)$ gets a cost of 0 if the $i+1$'th letter of the first word is the same as the $j+1$'th letter of the second word.

A shortest path from $(0, 0)$ to (m, n) represents an optimal solution to the problem, and could be found using the shortest path program.

6. (15 points) You have n products that need to be built. You also have m machines; any machine can build any product, but some machines are more efficient at certain tasks than others, and so the costs differ. More precisely, if we build product $j \in \{1, 2, \dots, n\}$ on machine $i \in \{1, 2, \dots, m\}$, the cost is c_{ij}; we are given all of these costs.

   Because of time limitations, we can build at most 5 products on each machine. A final complication is that there is a startup cost: if we decide to turn on a machine, we pay K. If we don't build any products on a machine, we don't pay the startup cost for that machine.

   Formulate an ILP that describes the problem of which machines to start, and which machine to use to build each product. As always, clearly explain the meaning of all variables and constraints.

   Hint: it will be useful to have variables x_{ij} which are 1 if task j is run on machine i, and 0 otherwise, for each machine $i \in \{1, 2, \dots, m\}$ and task $j \in \{1, 2, \dots, n\}$. You will need other variables as well.

Solution: Let x_{ij} be 1 if task j is run on machine i, and 0 otherwise. Let y_i be 1 if machine i is turned on, and 0 otherwise.

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} K y_i + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{m} x_{ij} = 1 \qquad \forall j \in \{1, 2, \dots, n\} \\
& \sum_{j=1}^{n} x_{ij} \le 5 y_i \qquad \forall i \in \{1, 2, \dots, m\} \\
& y_i \in \{0, 1\} \qquad \forall i \in \{1, 2, \dots, m\} \\
& x_{ij} \in \{0, 1\} \qquad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}
\end{aligned}
$$