Exam Operations Research
CALCULATOR IS NOT ALLOWED

Construction of the exam

This exam consists of 6 exercises, from which in total 80 points can be obtained.

Exercise	*a*	*b*	*c*
Exc. 1	5	10	5
Exc. 2	10	-	-
Exc. 3	5	5	5
Exc. 4	10	5	-
Exc. 5	10	-	-
Exc. 6	10	-	-

Exam mark

$$\frac{\text{total number of points}}{8}$$

Final course mark

$$\frac{1}{4}\text{Instruction tests} + \frac{3}{4}\text{Exam mark}$$

Exercise 1
Consider the LP-problem

$$\begin{aligned} \max \quad & 6x_1 - 3x_2 + 5x_3 \\ \text{s.t.} \quad & 4x_1 + 4x_2 + x_3 \leq 20 \\ & x_1 + x_2 + 3x_3 \leq 4 \\ & 2x_1 - x_2 + x_3 \leq 9 \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{1}$$

(a) Translate this LP-problem into standard basis form. Give the starting solution for the simplex algorithm and apply *one* iteration to get from the starting solution to an improved basic feasible solution. Expose and argue precisely how you find this improved basic feasible solution.

(b) What are the new basic variables? What are their values and what is the new objective value. How can you read from the new simplex tableau if the new solution is the optimal one? Is this the case here?

(c) Formulate the dual LP-problem. Without solving the dual, give the optimal dual solution and the optimal dual objective value.

Answers Exercise 1

(a) Standard form:

$$\max z = 6x_1 - 3x_2 + 5x_3$$

$$\begin{array}{lccccccccccccc} \text{onder} & 4x_1 & + & 4x_2 & + & x_3 & + & x_4 & & & & & = & 20 \\ & x_1 & + & x_2 & + & 3x_3 & & & + & x_5 & & & = & 4 \\ & 2x_1 & - & x_2 & + & x_3 & & & & & + & x_6 & = & 9 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & \geq & 0 \end{array}$$

Starting solution: $(0, 0, 0, 20, 4, 9)$ with objective value $z = 0$

Iteration 1:

Entering the basis: $x_1 = \min(\frac{20}{4}, \frac{4}{1}, \frac{9}{2}) = 4$. Leaving the basis: x_5. The new tableau becomes:

1	0	9	13	0	6	0	24
0	0	0	−11	1	−4	0	4
0	1	1	3	0	1	0	4
0	0	−3	−5	0	−2	1	1

(b) The new basic variables are x_1, x_4 and x_6 with values

$$x_1 = 4$$
$$x_4 = 4$$
$$x_6 = 1$$

and objective value $z = 24$.
Since all reduced coefficients (in the top-row) are positive we conclude that the optimal solution has been found.

(c) Dual problem:

$$\min w = 20y_1 + 4y_2 + 9y_3$$

$$\begin{array}{lcccccccc} \text{s.t.} & 4y_1 & + & y_2 & + & 2y_3 & \geq & 6 \\ & 4y_1 & + & y_2 & - & y_3 & \geq & -3 \\ & y_1 & + & 3y_2 & + & y_3 & \geq & 5 \\ & y_1, & & y_2, & & y_3 & \geq & 0 \end{array}$$

The optimal values of the dual variables can be read from the reduced objective coefficients of the slack variables in the optimal primal table. Thus: $y_1 = 0$, $y_2 = 6$, $y_3 = 0$ with objective value 24.

Exercise 2. *Please notice that for correct ingredients of the model in this exercise you can already gain points.*
A group of N general physicians (GPs) has decided to collaborate. However, they think it is important that every patient is assigned a GP who is his/her primary contact. They agree to reassign the whole set of M patients. The past records of each patient i

give an estimate t_i of the number of hours per year he/she requires consultation. In the reassignment they like to take distance between patient addresses and the GP's location into account: the distance between GP j and the address of patient i is d_{ij} $[i = 1, 2, ..., M; j = 1, ..., N]$. GP j has specified that he prefers to have a set of patients whose estimated consultancy requirements together do not exceed T_j hours per year, $[j = 1, ..., N]$. But patients have some influence. Each patient i has given a list S_i of GPs to whom they do *not* want to be assigned, $[i = 1, ..., M]$. Finally, patients 1 and 2 are a couple who have been promised to become assigned to the same GP. The GPs wish to make an assignment of the patients such that the total distance between the patient's addresses and the GPs assigned to them is minimized.

Formulate this problem as an integer linear programming problem.

Answer Exercise 2

- Introduce decision variables:
  x_{ij} having value 1 if patient i is assigned to GP j, and 0 otherwise.

- The objective function:

$$\min \sum_{i=1}^{M} \sum_{j=1}^{N} d_{ij} x_{ij}$$

- Each patient is assigned exactly once:

$$\sum_{j=1}^{N} x_{ij}, \quad i = 1, \ldots, M$$

- Time constraint for GP j:

$$\sum_{i=1}^{M} t_i x_{ij} \leq T_j \quad j = 1, \ldots, N$$

- Constraint for Non-assignments:

$$\sum_{j \in S_i} x_{ij} = 0, \quad i = 1, \ldots, M$$

  or

$$x_{ij} = 0, \text{ if } j \in S_i, \quad i = 1, \ldots, M$$

- constraint for couple 1,2

$$x_{1j} = x_{2j}, \quad j = 1, \ldots, N$$

- binary variables

$$x_{ij} \in \{0, 1\}, \quad i = 1, \ldots, M, \ j = 1, \ldots, N$$

Exercise 3.
Consider the following 0-1 knapsack problem.

$$\begin{array}{rl} \max & 5x_1 + 12x_2 + 9x_3 + 2x_4 \\ s.t.: & 2x_1 + 5x_2 + 4x_3 + 1x_4 \leq 10 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{array}$$

(a) Give the unique solution of the LP-Relaxation of this problem. Also give the corresponding objective value of the LP-Relaxation.

(b) Solve the 0-1 knapsack problem by Branch-and-Bound to find the optimal solution and the corresponding optimal value.

(c) Solve the *Knapsack* problem obtained by replacing the 0-1 constraints on the variables by non-negative integrality constraints.

Answers Exercise 3

(a) Order the items on non-increasing profit per unit weight: 1,2,3,4.
The greedy algorithm yields then $x_1 = 1$, $x_2 = 1$, $x_3 = \frac{3}{4}$, $x_4 = 0$ with value $23\frac{3}{4}$

(b) Just a sketch of the solution without pictures of the search tree: Branching on x_3 yields a solution of the LP-relaxation for $x_3 = 0$, having $x_1 = x_2 = x_4 = 1$ with value 19. This is a feasible integral solution, hence this node can be pruned on criterion P1 and we have already solution with value 19.
For $x_3 = 1$ we get $x_1 = 1$, $x_2 = \frac{4}{5}$, $x_4 = 0$ with value $23\frac{3}{5}$.
We branch further on the second node with $x_2 = 0$ and $x_2 = 1$. The first one, $x_2 = 0$ gives solution of the LP-relaxation of $x_1 = 1$,$x_2 = 0$, $x_3 = 1$, $x_4 = 1$ with value 16. Again a feasible solution, this time with value 16, which is pruned by criterion P1. The solution is worse than the one we had.
The node with $x_2 = 1$ has optimal solution of the LP-relaxation $x_1 = \frac{1}{2}$, $x_2 = x_3 = 1$, $x_4 = 0$ with value $23\frac{1}{2}$. This cannot be pruned.
Therefore, we branch further on x_1. The node with $x_1 = 0$ gives optimal solution of the LP-relaxation $x_1 = 0$, $x_2 = x_3 = x_4 = 1$, an integer solution with value 23. THis is the best solution found so far. The node itself is pruned by P1.
But now, all other open nodes can be pruned based on criterion P2.

(c) Just a sketch of the solution: The LP-relaxation has optimal solution $x_1 = 5$, and $x_2 = x_3 = x_4 = 0$ with value 25. This is a feasible integral solution, hence this node can be pruned on criterion P1 and we have found the optimal solution in one step.

Exercise 4.
Consider the network in the figure. The number next to an arrow indicates the capacity of that arrow.

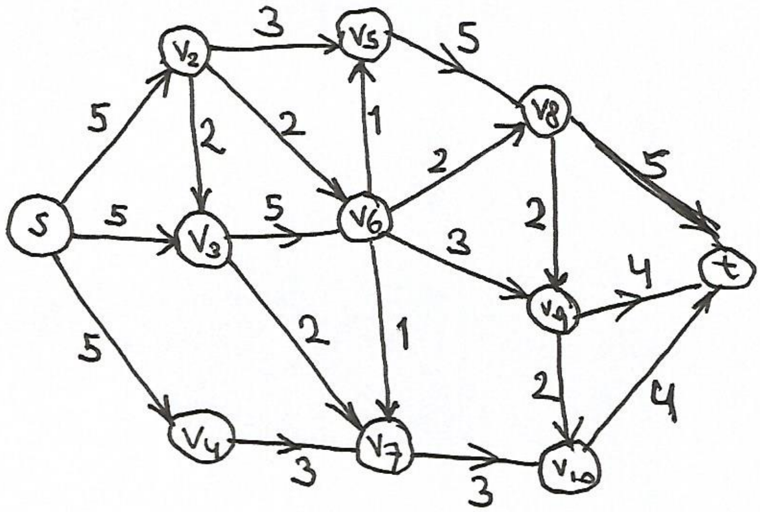
(a) Find the maximum flow from s to t using Ford-Fulkerson. Show clearly which flow augmenting paths you choose in each iteration.

(b) Using the residual network in the optimum, find the minimum s-t-cut. Clearly show how you find this minimum cut.

Answers Exercise 4

(a) Send flow over $s, 2, 5, 8, t$ with bottleneck capacity 3;
Send flow over $s, 3, 6, 8, t$ with bottleneck capacity 2;
Send flow over $s, 3, 6, 9, t$ with bottleneck capacity 3;
Send flow over $s, 4, 7, 10, t$ with bottleneck capacity 3;
Send flow over $s, 2, 6, 5, 8, 9, t$ with bottleneck capacity 1;
From here, no flow augmenting path can be found that uses only forward arrows. Also in the residual network the is no s, t-path. The optimal flow is 12.

(b) In the residual network (which I don't draw), the vertices $s, 2, 3, 4, 6, 7$ are still reachable from s. The total capacity of the cut is
$c_{25} + c_{65} + c_{68} + c_{69} + c_{7,10} = 3 + 1 + 2 + 3 + 3 = 12$


s
v2
v3
v4
v5
v6
v7
v8
v9
v10
t
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3
2
2
5
3
1
1
2
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2
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5
2
2
4
4

Exercise 5.
Delete the directions on the arcs in the figure of **Exercise 4**, making them undirected edges. Interpret the numbers in the figure as the costs of the corresponding edges. Find the Minimum Spanning Tree in the graph using your preferred method.
Answer Exercise 5

The easiest to write out is Kruskal. We order the edges on increasing weight:
$(5,6),(6,7),(2,6),(2,3),(3,7),(6,8),(8,9),(9,10),(2,5),(4,7),(7,10),(6,9),(9,t),(10,t),(s,2),(s,3),(s$
Then we select:

$$(5,6),(6,7),(2,6),(2,3)$$

The next one $(3,7)$ would create a cycle. Thus we select

$$(6,8),(8,9),(9,10)$$

The next one $(2,5)$ would create a cycle. Thus we select

$$(4,7)$$

The next ones $(7,10),(6,9)$ would create a cycle. Thus we select

$$(9,t)$$

The next one $(10,t)$ would create a cycle. Thus we select

$$(s,2).$$

Since we have 11 vertices and we have now 10 edges we stop the search and output the tree

$$(5,6),(6,7),(2,6),(2,3),(6,8),(8,9),(9,10),(4,7),(9,t),(s,2)$$

with total weight 24.

Exercise 6. *Please notice that for correct ingredients of the model in this exercise you can already gain points.*
Kylie is planning a world-trip and likes to visit a total of N different countries, the countries $1,\ldots,N$, and eventually return to the Netherlands (= country 0). She starts also in the Netherlands and the order in which she will visit the N countries is still to be determined. If Kylie flies from country i to country k it will cost her an amount of v_{ik} euro, $i=0,\ldots,N,\ k=0,\ldots,N$. Moreover, she does not want to visit any country twice or more. **Notice**: the $(N+1)$-th country she will visit is fixed, the Netherlands.

How will her optimal world-trip be (including the return to the Netherlands), such as to minimize total travel expenses?

Formulate this problem as a dynamic programming problem. To do so, describe the states and the interpretation of the value function in words. Give the recursion and the starting conditions and describe what needs to be computed in terms of the value function. ***Hint:*** *Start with stage j.*

Answer Exercise 6

- Stage j corresponds to the j-th country that she visits, having already decided on the visits $1, \dots, j-1$ and is currently to decide on the j-th country to go to.
- A state is defined by the country i where she is and the countries S still to be visited at the beginning of the stage.
- The decision is which next country $k \in S$ to go to.
- The direct cost of a decision k is v_{ik}
- State transition $S \to S \setminus k$.
- The value function $f_j(i, S)$ is the minimum total cost if we are to decide on the j-th country to go to, being in country i and still to visit countries S.
- The recursion is given by

$$f_j(i, S) = \min_{k \in S} \{v_{ik} + f_{j+1}(k, S \setminus k)\}$$

- To be computed: $f_0(0, \{1, \dots, N\})$
- Starting constraints: $f_{N+1}(j, N+1) = 0$ and $f_{N+1}(j, S) = \infty 0$, $\forall j, S$.