Exam Operations Research
CALCULATOR IS NOT ALLOWED

Construction of the exam

This exam consists of 6 exercises, from which 100 points can be obtained in total. The division of the points over the various parts is given in the table.

Exercise	*a*	*b*	*c*
Exc. 1	10	10	5
Exc. 2	15	-	-
Exc. 3	5	15	-
Exc. 4	10	5	-
Exc. 5	10	-	-
Exc. 6	15	-	-

Exam mark

$$\frac{\text{total number of points}}{10}$$

Final course mark

$$\tfrac{1}{4}\text{Instruction mark} + \tfrac{3}{4}\text{Exam mark}$$

Exercise 1

$$\begin{array}{llrcrcrcr} \max & Z = & 5x_1 & - & x_2 & + & 4x_3 & & \\ \text{s.t.} & & 2x_1 & + & 2x_2 & - & x_3 & \leq & 5 \\ & & 3x_1 & - & 2x_2 & + & 3x_3 & \leq & 14 \\ & & & & & & \multicolumn{3}{r}{x_1, x_2, x_3 \geq 0.} \end{array}$$

(a) Translate this LP-problem into standard basis form. Give the starting solution for the simplex algorithm and apply *one* iteration to get from the starting solution to an improved basic feasible solution. Show precisely how you find this improved basic feasible solution.

(b) Formulate the dual LP-problem.

(c) The following simplex tableau displays the optimal solution of the primal problem.

Basic	z	x_1	x_2	x_3	s_1	s_2	*value*
(Z)	1	2.75	0	0	1.25	1.75	30.75
(x_2)	0	2.25	1	0	0.75	0.25	7.25
(x_3)	0	2.50	0	1	0.50	0.50	9.50

Without solving the dual, give the optimal dual solution and the optimal dual objective value.

Answers Exercise 1

Answer 1a. First we bring the problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcl}
\max & Z & = & 5x_1 & - & x_2 & + & 4x_3 & & & & & & \\
\text{s.t.} & & & 2x_1 & + & 2x_2 & - & x_3 & + & s_1 & & & = & 5 \\
 & & & 3x_1 & - & 2x_2 & + & 3x_3 & + & & + & s_2 & = & 14 \\
 & & & & & & & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, s_1, s_2 \geq 0.}
\end{array}
$$

The starting tableau is

(z)	1	-5	1	-4	0	0	0
(s_1)	0	2	2	-1	1	0	5
(s_2)	0	3	-2	3	0	1	14

Choose x_1 as entering variable. The minimum ratio test takes $\min\left\{\frac{5}{2}, \frac{14}{3}\right\} = \frac{5}{2} \Rightarrow s_1$ is leaving variable. The new tableau becomes

(z)	1	0	6	$-6\frac{1}{2}$	$\frac{5}{2}$	0	$12\frac{1}{2}$
(x_1)	0	1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$2\frac{1}{2}$
(s_2)	0	0	-5	$\frac{9}{2}$	$-\frac{3}{2}$	1	$6\frac{1}{2}$

Answer 1b. The dual problem is

$$
\begin{array}{lrcrcrcr}
\min & W & = & 5y_1 & + & 14y_2 & & \\
\text{s.t.} & & & 2y_1 & + & 3y_2 & \geq & 5 \\
 & & & 2y_1 & - & 2y_2 & \geq & -1 \\
 & & & -y_1 & + & 3y_2 & \geq & 4 \\
 & & & & & \multicolumn{3}{r}{y_1, y_2 \geq 0.}
\end{array}
$$

Answer 1c. The dual optimal solution is given by the reduced objective coefficients of the starting variables (the slack variables in this case) plus their original objective coefficients (thus 0 in this case). Hence $y_1 = 1.25$ and $y_2 = 1.75$. The optimal value is according to the strong duality theorem equal to 30.75. For verification of the optimal value, the dual solution has value $5 \times \frac{5}{4} + 14 \times \frac{7}{4} = \frac{123}{4} = 30\frac{3}{4}$.

Exercise 2. *Please notice that for correct parts of the model in this exercise you can already gain points.* A group of 3 general physicians, GP1,GP2 and GP3, share their practice. Every day, at the start of the day, the patients who have asked for a consult are assigned to the 3 GPs. Each patient has indicated if he requires a short consult (15 minutes) or a long consult (30 minutes). This week there are 20 patients, numbered $1, 2, \ldots, 20$. The first 7 patients have indicated to need a long consult. All the others require a short consult.

The GPs would like to find an assignment of the patients such that the total amount of time they will be busy is as equal as possible. But they are a very social team, and GP3 joined the practice just a month ago, and it is her first job as an independent GP. This means that she needs twice as much time for the patients as is indicated: thus, a short consult takes her 30 minutes and a long consult takes her 1 hour.

For the same reason there are some patients that should not be assigned to GP3. These are the patients numbered 1, 5 and 18. Moreover, GP2 does not like to get more than 2 long consult patients. GP1 gives the restriction that if he gets assigned patient 13 then he does not want to see patient 14. Finally, patients 9 and 10 form a couple and should *not* be assigned to the same GP.

Formulate this problem as an integer linear optimization problem.

Answer 2. Introduce binary variables (2pt) x_{ij} that have value 1 if patient j is assigned to GPi, $i = 1, 2, 3$, $j = 1, \ldots, 20$. I also introduce the variable T_i for the total time that GPi will be busy, $i = 1, 2, 3$. There is some choice in the objective, since "as equal as possible" can be formulated in various ways. For example it can be modelled as minimizing the busy time of the GP who is busy longest, or maximizing the busy time of the GP who is busy shortest, or minimizing the sum of the pairwise differences between the busy times of the GPs. Let me take the first one, for which I introduce yet another auxiliary variable w and define objective (3pts)

$$\min w$$

subject to:

$$w \geq T_1$$
$$w \geq T_2$$
$$w \geq T_3$$

Restrictions from definitions of T_i's (3pts)

$$T_1 = \sum_{j=1}^{7} 30x_{1j} + \sum_{j=8}^{20} 15x_{1j}$$

$$T_2 = \sum_{j=1}^{7} 30x_{2j} + \sum_{j=8}^{20} 15x_{2j}$$

$$T_3 = \sum_{j=1}^{7} 60x_{3j} + \sum_{j=8}^{20} 30x_{3j}$$

Every patient needs to be assigned (2pts)

$$x_{1j} + x_{2j} + x_{3j} = 1, \;\; j = 1, \ldots, 20$$

Extra restrictions because of GP-patient combinations (1pt each)

$$x_{31} = 0, x_{35} = 0, x_{3,18} = 0$$

$$\sum_{j=1}^{7} x_{2j} \leq 2$$

$$x_{1,14} \leq 1 - x_{1,13}$$

$$x_{i9} + x_{i,10} \leq 1, \;\; i = 1, 2, 3$$

Binarity of the variables (1pt)

$$x_{ij} \in \{0, 1\}, \;\; i = 1, 2, 3, j = 1, \ldots, 20$$

Exercise 3.
Consider the following 0-1 knapsack problem.

$$\begin{array}{ll} \max & 7x_1 + 8x_2 + 3x_3 + 10x_4 \\ \text{s.t.}: & 4x_1 + 5x_2 + x_3 + 5x_4 \leq 9 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{array}$$

(a) Determine the unique solution of the LP-Relaxation of this problem. Also give the corresponding optimal objective value of the LP-Relaxation.
Answer 3a. Order the items on non-increasing profit per unit weight: 3,4,1,2.
The greedy algorithm yields then $x_3 = 1$, $x_4 = 1$, $x_1 = \frac{3}{4}$, $x_2 = 0$ with value $18\frac{1}{4}$

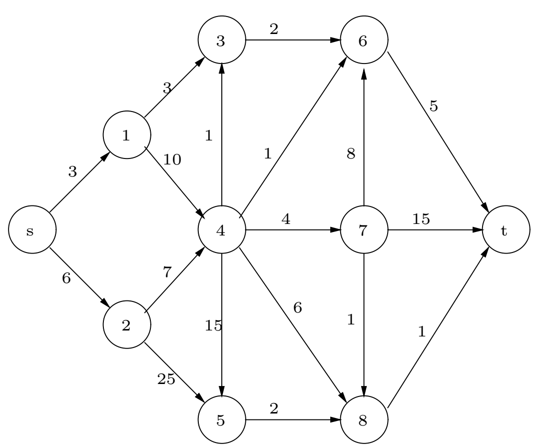
(b) Solve the 0-1 knapsack problem using Branch&Bound. Clearly indicate in which order you compute the nodes of the search tree and where you prune the search tree, and based on which pruning criterion.

Answer 3b. Just a sketch of the solution without pictures of the search tree: Branching on x_1 yields a solution of the LP-relaxation for $x_1 = 0$, having $x_3 = x_4 = 1$ en $x_2 = \frac{3}{5}$ with value $17\frac{4}{5}$. Hence, rounded down this yields 17 as upper bound. The solution itself rounded down gives $x_1 = x_2 = 0$ and $x_3 = x_4 = 1$ with value 13. For $x_1 = 1$ we get $x_3 = 1$, $x_4 = \frac{4}{5}$, $x_2 = 0$ with value 18. (Take care! Although the value of the solution is here integer, the solution itself is not!). We branch further on the second node with $x_4 = 0$ and $x_4 = 1$. The first one, $x_4 = 0$ gives solution of the LP-relaxation of $x_1 = 1$,$x_4 = 0$, $x_3 = 1$, $x_2 = \frac{4}{5}$ with value $16\frac{2}{5}$. The node with $x_4 = 1$ happens to have as optimal solution of the LP-relaxation the integer solution $x_1 = x_4 = 1$, $x_2 = x_3 = 0$ with value 17. Therefore, this node can be pruned because of criterion P1. But now, all other open nodes can be pruned based on criterion P2.

Exercise 4.
Consider the max-flow problem on the network in the figure. The number next to an arrow gives the capacity of that arrow.

(a) Find the maximum flow from s to t using Ford-Fulkerson. Show clearly which flow augmenting paths you choose in each iteration.
Answer.
Send flow over $s, 2, 4, 7, t$ with bottleneck capacity 4;
Send flow over $s, 1, 3, 6, t$ with bottleneck capacity 2;
Send flow over $s, 2, 4, 6, t$ with bottleneck capacity 1;
Send flow over $s, 2, 5, 8, t$ with bottleneck capacity 1;
From here, no flow augmenting path can be found that uses only forward arrows. Neither in the residual network (which I don't draw here) there is an s-t-path. Hence the optimal flow value is 8.

(b) Using the residual network in the optimum, find the minimum s-t-cut.
Answer. In the residual network (which I don't draw), the vertices $s, 1, 2, 3, 4, 5, 8$ are still reachable from s. The total capacity of the cut is $c_{36} + c_{46} + c_{47} + c_{8t} = 2 + 1 + 4 + 1 = 8$



Exercise 5.
Undirect the arrows in the figure of **Exercise 4** and interpret the numbers in the figure as weights of the corresponding edges. Find the minimum spanning tree in the resulting graph. State whether you use Prim's or Kruskal's algorithm and clearly show how you use the algorithm.

Answer Exercise 5

The easiest to write out is Kruskal. We order the edges on increasing weight:
$(3,4),(4,6),(7,8),(8,t),(5,8),(3,6),(s,1),(1,3),(4,7),(6,t),(s,2),(4,8),(2,4),(6,7),(1,4),(4,5),(7,t),(2,5)$.
Then we select:

$$(3,4),(4,6),(7,8),(8,t),(5,8)$$

The next one $(3,6)$ would create a cycle. Thus we select

$$(s,1),(1,3),(4,7)$$

The next one $(6,t)$ would create a cycle. Thus we select

$$(s,2)$$

Since we have 10 vertices and we have now 9 edges we stop the search and output the tree

$$(3,4),(4,6),(7,8),(8,t),(5,8),(s,1),(1,3),(4,7),(s,2)$$

with total weight 22.

Opgave 6.
A manufacturer of bicycles is planning his production of racing bicycles for the 6 months of the season. Outside of the season there is no demand from the retailers for racing bicycles. Market data from the past have given good estimates of what the expected demand will be. They are given in the table below. The production cost differs per month. The unit production cost are also given in the table. If racing bicycles are produced in a month, they are produced in one large batch because the production of racing bicycles requires a different setting of most of the machine park than the production of other bicycles. The cost of setting up the production of a batch of racing bicycles is 1000 euro. This setup cost is the same in each month. Finally, for each of the racing bicycles remaining in stock at the end of a month a holding cost of 20 euro is paid, also equal over all 6 months.

Month	demand	unit cost
April	700	150
May	1000	100
June	900	130
July	500	170
August	700	170
Sept.	300	100

Formulate this problem as a dynamic programming problem. To do so, describe the states and the interpretation of the value function in words. Give the recursion and the starting conditions and describe what needs to be computed in terms of the value function. ***Hint:*** *Start with defining the stage.*

Answer Exercise 6

Let me number the months, $1 = April$, $2 = May$,...,$7 = October$, write d_j for the demand in month j, and c_j for the unit production cost in month j. Later I use p_j for the production in month j.

- For a stage we take the month: $1 = April$, $2 = May$,...,$7 = October$, at stage j we have already decided on the production of bicycles in the months $1, \ldots, j-1$ and are to decide on the production in month j.
- As a state I define b, the number of racing bicycles in stock at the beginning of the month.
- The decision is how much to produce in the month j
- The direct cost of a decision is
  - $20(b - d_j)$ if nothing is produced in month j, which is feasible only if $b \geq d_j$.
  - $1000 + c_j p_j + 20(b + p_j - d_j)$ if $p_j > 0$ is produced
- The value function $f_j(b)$ is the minimum cost if we are to decide about production in month j and have a stock of b at the start of month j and we still have to produce for months $j, j+1, \ldots, 6$.
- The recursion is given by

$$f_j(b) = \min \left\{ 20(b - d_j) + f_{j+1}(b - d_j), \min_{p_j | b + p_j \geq d_j} \{1000 + c_j p_j + 20(b + p_j - d_j) + f_{j+1}(b + p_j - d_j)\} \right\}$$

- To be computed: $f_1(0)$
- Starting constraints: $f_7(0) = 0$ and $f_7(b) = M > 0 \ \forall b > 0$, where in fact choosing $M = 1$ is already sufficient. We add to the starting conditions that $f_j(b) = \infty$ if $b < 0$, for all j, to enforce that decision $p_j = 0$ is only made when $b > d_j$.