Dept. Computer Science	Networks and Graphs
VU University Amsterdam	29.05.2013

BE SURE THAT YOUR HANDWRITING IS READABLE

Part I

1a Let G denote a simple graph with n vertices and m edges. For each of the following mathematical statements, (1) translate the statement into common English and (2) tell whether it is true or false.

1. $G[E(G)] \subseteq G$.
2. $\forall u \in V(G) : \delta(u) \geq \min\{\delta(v)|v \in V(G)\}$
3. G is connected $\Rightarrow n \leq m$
4. $\exists H, H' \subseteq G : G[V(H) \cup V(H')] = K_n$.
5. $\exists u, v \in V(G) : \nexists(u, v)$-path $\Rightarrow \omega(G) > 1$

10pt

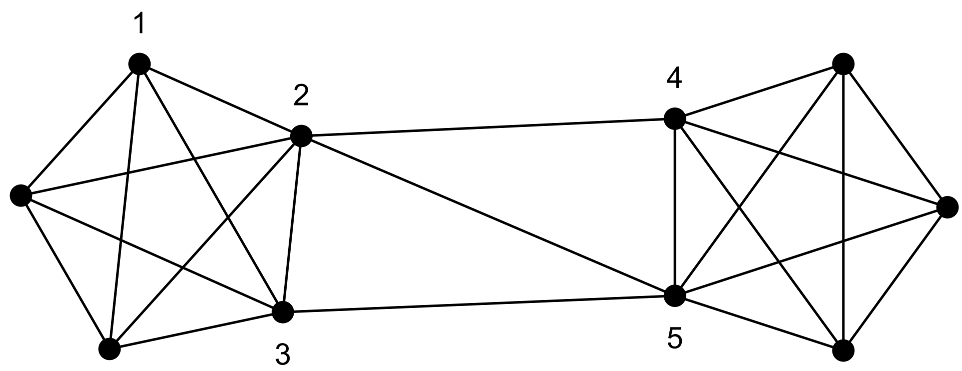
1. *The graph induced by the edgeset of G is a subgraph of G. True.*
2. *Each vertex in G has a degree that is greater or equal to the minimal degree. True.*
3. *If G is connected, then the number of vertices of G is less or equal to the number of edges. False.*
4. *There exists two subgraphs of G such that the graph induced by the joint set of their vertices is the complete graph on n vertices. False.*
5. *If there are two distinct vertices in G that are not connected through a path, then G will consist of more than one component. True.*

2a Prove that for any simple graph G, $\lambda(G) \leq \min\{\delta(v)|v \in V(G)\}$. *5pt*

Consider a vertex u with minimal degree. If we remove the $\delta(u)$ edges incident with u, then u will become isolated, and certainly the resulting graph will have at least one more component then it had before (namely the one consisting only of u).

2b Construct a graph for which $\kappa(G) < \lambda(G) < \min\{\delta(v)|v \in V(G)\}$. *5pt*

Consider the following graph. Clearly, $\delta(1) = 4$ and is also the minimum vertex degree of G. Furthermore, the set $\{\langle 2,4\rangle, \langle 2,5\rangle, \langle 3,5\rangle\}$ forms a minimal edge cut of size 3, whereas the set of vertices $\{2,5\}$ forms a minimal vertex cut of size 2.



3a Prove that for any connected simple planar graph with $n \geq 3$ vertices and m edges, $m \leq 3n-6$. *10pt*

Consider a region f in any plane graph of G. For any interior region, let $B(f)$ denote the number of edges by which f is enclosed. $B(f) \geq 3$ for any interior region. With $n \geq 3$ we also have that the exterior region is "bounded" by at least 3 edges. As a consequence, if there are a total of r regions, then $\sum B(f) \geq 3r$. Note that $\sum B(f)$ counts every edge in G once or twice, and hence $\sum B(f) \leq 2m$, so that $3r \leq \sum B(f) \leq 2m$, and thus $r \leq \frac{2}{3}m$. Because we know that $m = n+r-2$, we know that $m \leq n + \frac{2}{3}m - 2$, and thus that $m \leq 3n-6$.

3b Prove that every connected, planar graph has a vertex with degree less or equal to five. *5pt*

For graphs with $n \leq 6$ vertices the statement is obviously true. Let G be a connected planar graph with $n > 6$ vertices and m edges. Assume all vertices have a degree higher than 5. Because $2m = \sum \delta(v) \geq 6n$, and $m \leq 3n-6$, we would have $6n \leq 6n-12$, which is impossible. Hence, not all vertices can have degree 5 or more.

4a Give definitions for (1) trail, (2) path, (3) Euler trail, and (4) Euler path. *4pt*

(1) A walk in which all edges are traversed at most once. (2) A trail in which all vertices are traversed at most once. (3) A trail in which all *edges of a graph are traversed exactly once. (4) A path in which* all *edges of a graph are traversed exactly once.*

4b Complete the following statement: "Graph G contains an Euler trail if and only if ..." *3pt*

G has exactly two vertices having odd degree.

4c Complete the following statement: "Graph G contains an Euler path if and only if ..." *3pt*

G has the form of a single (u,v)-path, with $u \neq v$.

5 Prove by induction that the number of edges m of the complete graph with n vertices is equal to $\frac{1}{2}n(n-1)$. *5pt*

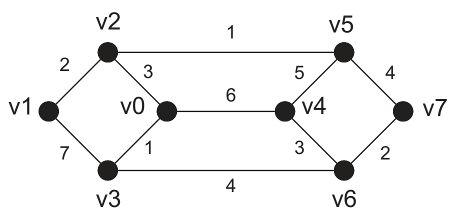
The statement is trivially true for $n = 1$: $m = 0 = \frac{1}{2} \cdot 1 \cdot 0$. Assume the statement holds for the complete graph with $k > 1$ vertices and consider K_{k+1}. Remove any vertex u, to obtain K_k, having, by induction, a total of $\frac{1}{2}k(k-1)$ edges. Vertex u was joined with k vertices, so that by removing u, we also removed k edges. This means that $|E(K_{k+1})| = k + \frac{1}{2}k(k-1) = k + \frac{1}{2}k^2 - \frac{1}{2}k = \frac{1}{2}k^2 + \frac{1}{2}k = \frac{1}{2}(k+1)k$, completing the proof.

Part II

6a What is the minimal number of rounds needed in the Bellman-Ford algorithm so that each node has found the shortest path to every other node? Be sure to explain your answer. *6pt*

In the worst case, the shortest path in terms of weights, will consist of the maximal number d of edges between any two vertices. As a consequence, in order for any two nodes to be sure that they have found the shortest path, at least $d/2$ rounds will be necessary (as path information grows from two sides). Any answer that mentioned diameter (which is getting very close), or better: longest path, was considered correct.

6b What is routing table of vertex $v5$ after three rounds of the Bellman-Ford algorithm? Explain your answer. *6pt*



The routing table will contain the information how to reach any other node at hop-distance 3, and at minimal path distance. If we look at the first three rounds, we'll see the following development.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
1:	∞	∞	$(1,v_2)$	∞	$(5,v_4)$	$(0,v_5)$	∞	$(4,v_7)$
2:	$(4,v_2)$	$(3,v_2)$	$(1,v_2)$	∞	$(5,v_4)$	$(0,v_5)$	$(6,v_7)$	$(4,v_7)$
3:	$(4,v_2)$	$(3,v_2)$	$(1,v_2)$	$(5,v_2)$	$(5,v_4)$	$(0,v_5)$	$(6,v_7)$	$(4,v_7)$

7a The clustering coefficient for a real-world network with 1000 vertices and 7500 edges is 0.1. Is this high? Explain your answer. *6pt*

We need to compare this to an Erdös-Rényi graph $ER(1000,p)$. The average degree in the real-world network can be computed as $\frac{1}{n}\sum\delta(v) = \frac{2m}{n} = 15$. We know that the average degree for an $ER(n,p)$ graph is equal to $p(n-1)$, meaning that we should compare our real-world network to a network for which $p(n-1) = 15$, and thus $p \approx 0.015$. This is also the clustering coefficient for the ER graph, and we conclude that 0.1 is indeed relatively high.

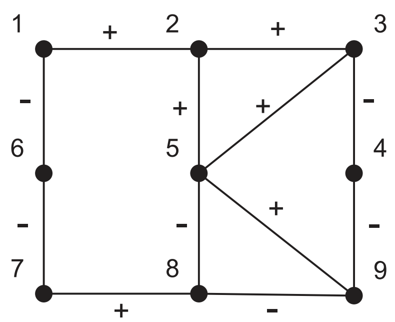
7b Explain how to construct a scale-free network. *8pt*

Start with a random network and then repeatedly add a vertex v, connecting it to m existing vertices, where the probabibily of joining v to u is proportional to the degree of u: the higher $\delta(u)$, the higher the probability of adding the link $\langle v,u\rangle$ (provided the link does not yet exist). Stop when the graph has n vertices.

7c Explain how to construct a Watts-Strogatz graph. *6pt*

Start with placing n nodes in a "circle," joining each node to its $k/2$ left-hand neighbors, and $k/2$ right-hand neighbors. Then, for each edge $\langle u,v\rangle$, rewire that edge with probability p to an edge $\langle u,w\rangle$ with w arbitrarily chosen and keeping the graph simple.

8a Compute the center of the following (signed) graph, as well as its clustering coefficient. *6pt*



Vertices 1,2,3,5,7,8,9 have (lowest) eccentricity 3, vertices 6,4 have 4. That means that the center consists of vertices 1,2,3,5,7,8,9. The clustering coefficient for all these vertices except 1,7 is $\frac{1}{3}$, the others have cc 0, meaning the the cc for the entire graph is equal to $(5\cdot\frac{1}{3}+4\cdot 0)/9 = \frac{5}{27}$.

8b Systematically check if the previous graph is balanced. *6pt*

The crux to your answer is that you show whether or not the set of vertices can be split into two subsets with negative-signed edges between the two sets, and positive-signed edges between nodes in either set. This is not the case: start with adding vertex 1 into V_0. Then add vertex 2 to V_0 and 6 to V_1. From vertex 2, add 3 and 5 to V_0; from 6 add 7 to V_0. From vertex 3 add 4 to V_1, and from vertex 5 add 9 to V_0 and 8 to V_1. We now have a positive-signed edge between V_0 and V_1: $\langle 7,8\rangle$, so the graph is not balanced.

8c Consider an affiliation network with adjacency matrix $\mathbf{AE}$, representing n_p people and n_e events, where $\mathbf{AE}[i,j] = 1$ if and only if person v_i participates in event e_j, and otherwise $\mathbf{AE}[i,j] = 0$. Explain what is meant by (1) $\sum_{k=1}^{n_e}(\mathbf{AE}[i,k]\cdot\mathbf{AE}[j,k])$ and likewise (2) $\sum_{k=1}^{n_a}(\mathbf{AE}[k,i]\cdot\mathbf{AE}[k,j])$. *4pt*

$\sum_{k=1}^{n_e}(\mathbf{AE}[i,k]\cdot\mathbf{AE}[j,k])$ *adds the number of events in which both v_i and v_j participate.* $\sum_{k=1}^{n_a}(\mathbf{AE}[k,i]\cdot\mathbf{AE}[k,j])$ *adds the number of participants in both event e_i as well as event e_j.*

9a Explain that the probability $\mathbb{P}[\delta(v) = k]$ in an $ER(n, p)$ graph is equal to $\binom{n-1}{k}\ p^k\ (1-p)^{n-1-k}$ *8pt*

There are a maximum of $n-1$ other vertices that can be a neighbor of v. As there are $\binom{n-1}{k}$ possibilities for choosing k different vertices to be adjacent to v, the probability of having any specific set of k edges connecting v is as given.

9b Compute the clustering coefficient for an $ER(n, p)$ graph. *6pt*

Every vertex with k neighbors can expect a total of $\binom{k}{2} \cdot p$ edges between those neighbors. Because the maximal number of edges between neighbors is $\binom{k}{2}$, the clustering coefficient for each vertex is exactly p.

Final grade: *(1) Add, per part, the total points. (2) Let T denote the total points for the midterm exam ($0 \leq T \leq 50$); D1 the total points for part I; D2 the total points for part II. The final number of points E is equal to* $\max\{T, D1\} + D2$.