vrije Universiteit *amsterdam*

School of Business and Economics

Exam:	Multivariate Econometrics
Code:	E_EORM_MVE
Examinator: Co-reader:	dr. H. Karabiyik dr. P. Gorgi
Date:	December 23, 2021
Time:	08:30
Duration:	2 hours and 45 minutes
Calculator allowed:	Yes
Graphical calculator allowed:	No
Number of questions:	4 (where each question consists of multiple parts)
Type of questions:	Open
Answer in:	English

Remarks:
- Read all questions carefully
- Motivate all your answers.

Credit score:	A score of 100 points counts for a grade 10 for the exam. Each question is 25 points.
Grades:	The grades will be made public within 10 working days after the exam.
Inspection:	TBA
Number of pages:	11 (including front page)

Good luck!

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Instructions

(i) All questions should be answered to get full points.

(ii) Each question is worth 25 points.

(iii) Read the instructions in the questions carefully.

(iv) Answer the questions as detailed as possible. Use mathematical expressions when necessary. You can use words when you cannot provide a formal mathematical answer to the questions.

(v) If a question is not clear to you, make your own assumptions to clarify the meaning of the question and then answer the question based on your assumptions.

(vi) See the back of this page for some standard results that you may make use of while answering the questions.

Some standard results

Suppose that the scalar process $\{z_t\}$ follows the following data generating process:

$$z_t = z_{t-1} + u_t,$$

where $z_0 = 0$ and u_t has the following properties:

(a) $u_t = \psi(L)\epsilon_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$ where $\sum_{j=0}^{\infty} j \cdot |\psi_j| < \infty$ and $\{\epsilon_t\}$ is an $i.i.d$ sequence with mean zero and variance σ_ϵ^2, and finite fourth moment;

(b) σ^2 denotes the long run variance of $\{u_t\}$ and σ_u^2 denotes the contemporaneous variance of $\{u_t\}$.

Note that under these assumptions z_t can be written as a partial sum as

$$z_t = \sum_{s=1}^{t} u_s.$$

Let $W(r)$ be a standard Brownian motion process associated with u_t. Then the following results hold:

(1) $T^{-1/2} \sum_{t=1}^{T} u_t \xrightarrow{d} \sigma W(1)$;

(2) $T^{-1} \sum_{t=1}^{T} u_t^2 \xrightarrow{p} \sigma_u^2$;

(3) $T^{-1} \sum_{t=1}^{T} z_{t-1} u_t \xrightarrow{d} \frac{1}{2}\sigma^2 \left[W(1)^2 - \frac{\sigma_u^2}{\sigma^2} \right]$;

(4) $T^{-3/2} \sum_{t=1}^{T} t u_{t-j} \xrightarrow{d} \sigma \left\{ W(1) - \int_0^1 W(r) dr \right\}$ for $j = 0, 1, \ldots$;

(5) $T^{-3/2} \sum_{t=1}^{T} z_{t-1} \xrightarrow{d} \sigma \int_0^1 W(r) dr$;

(6) $T^{-2} \sum_{t=1}^{T} z_{t-1}^2 \xrightarrow{d} \sigma^2 \int_0^1 W(r)^2 dr$;

(7) $T^{-5/2} \sum_{t=1}^{T} t z_{t-1} \xrightarrow{d} \sigma \int_0^1 r W(r) dr$;

(8) $T^{-3} \sum_{t=1}^{T} t z_{t-1}^2 \xrightarrow{d} \sigma^2 \int_0^1 r W(r)^2 dr$;

(9) $T^{-(v+1)} \sum_{t=1}^{T} t^v \to 1/(v+1)$ for $v = 0, 1, \ldots$;

(10) Suppose that the DGP of another time series process y_t follows the model

$$y_t = y_{t-1} + e_t,$$

where $y_t = 0$ and e_t satisfy the same assumptions as **(a)** and has a long run variance σ_e^2 and $W_e(r)$ is a standard Brownian motion processs associated with e_t, then

$$T^{-2} \sum_{t=1}^{T} z_t y_t \xrightarrow{d} \sigma \sigma_e \int_0^1 W(r) W_e(r) dr.$$

Question 1: Conceptual Questions (25 points out of 100 points)

Below you will find **3** statements. All these are related to the concepts/techqniques that have been discussed during the lectures. Some of these statements are correct, some are wrong, some need further clarification. You need to provide a brief, to the point answer that would contain **(i) short explanations/definitions of the concepts mentioned in the statement, (ii) your judgement about the statement about whether it is correct/wrong/unclear/incomplete, and an explanation of your judgement (iii) a correction of the statement.** The concepts that you need to explain and define are written in *italics*. A formal answer using mathematics is possible, sometimes very useful but not always necessary.

(a) (5 points) Consider a *stable VAR(1) process*. The j^{th} order *autocovariance* of this process is equal to its 1^{st} order autocovariance, when $j \neq 1$.

(b) (10 points) Consider the time series x_t with the DGP

$$x_t = \lambda x_{t-1} + \varepsilon_t,$$

where $\lambda = 1$ and $\varepsilon_t \sim i.i.d(0, \sigma_\varepsilon^2)$. This implies that x_t has a *unit root.* In this case, the OLS estimator of λ is $\sqrt{T}$-consistent and it is asymptotically normally distributed.

(c) (10 points) For panel data variables $y_{i,t}$ and $\mathbf{x}_{i,t}$ that are observed for N cross section units, over T time periods, a *homogeneous panel data model* can be written as

$$y_{i,t} = \boldsymbol{\beta}'\mathbf{x}_{i,t} + u_{i,t}.$$

When $Cov(u_{i,t}u_{j,t}) \neq 0$ for $i = j$, we say that i and j are cross-sectionally dependent. If this is the case for all i and j, the *pooled OLS estimator* of $\boldsymbol{\beta}$ will be inconsistent. If we estimate $\boldsymbol{\beta}$ by using the *mean group estimator* the estimator will be consistent.

Question 2: Modeling and stationarity (25 points out of 100 points)

(a) (15 points) Consider three processes y_t, x_t and z_t with the DGPs

$$\Delta y_t = \alpha_1(y_{t-1} - \beta_1 x_{t-1} - \beta_2 z_{t-1}) + \gamma_{11}\Delta y_{t-1} + \gamma_{13}\Delta z_{t-1} + u_{y,t},$$
$$\Delta x_t = \alpha_2(y_{t-1} - \beta_1 x_{t-1} - \beta_2 z_{t-1}) + \gamma_{21}\Delta y_{t-1} + \gamma_{22}\Delta x_{t-1} + u_{x,t},$$
$$\Delta z_t = \alpha_3(y_{t-1} - \beta_1 x_{t-1} - \beta_2 z_{t-1}) + \gamma_{32}\Delta x_{t-1} + \gamma_{33}\Delta z_{t-1} + u_{z,t},$$

where

$$\begin{pmatrix} u_{y,t} \\ u_{x,t} \\ u_{z,t} \end{pmatrix} \sim MVN\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{pmatrix}\right].$$

Now we define

$$\mathbf{w}_t = \begin{pmatrix} y_t \\ x_t \\ z_t \end{pmatrix}.$$

(i) Write the vector error correction model (VECM) for $\Delta \mathbf{w}_t$.

(ii) Obtain the vector autoregression representation (VAR) for $\mathbf{w}_t$.

(iii) Assume that $\alpha_2 = \alpha_3 = \gamma_{21} = \gamma_{32} = 0$ and $|\gamma_{22}| < 1$, $|\gamma_{33}| < 1$. Establish and discuss the order of integration d, i.e. I(d) of y_t, x_t and z_t and the cointegration relations between them (as well as the number and the form of the cointegrating vectors, if any).

(iv) Derive the conditional error correction model (CECM) for y_t given x_t and its past (you can ignore the model for z_t).

(b) (10 points) For panel data variables $y_{i,t}$ and $\mathbf{x}_{i,t}$ that are observed for N cross section units, over T time periods, a homogeneous panel data model can be written as

$$y_{i,t} = \boldsymbol{\beta}'\mathbf{x}_{i,t} + u_{i,t},$$

where

$$u_{i,t} = \boldsymbol{\lambda}_i'\mathbf{f}_t + \varepsilon_{i,t},$$

where $\mathbf{f}_t$ is a $k \times 1$ vector of unobserved global shocks, $\boldsymbol{\lambda}_i$ is the $k \times 1$ vector of factor loadings and $\varepsilon_{i,t}$ is the idiosyncratic error. Furthermore, we assume that the $m \times 1$ vector of $\mathbf{x}_{i,t}$ has the following DGP:

$$\mathbf{x}_{i,t} = \boldsymbol{\Lambda}_i\mathbf{f}_t + \boldsymbol{\nu}_{i,t},$$

where $\boldsymbol{\Lambda}_i$ is the $m \times k$ matrix that contains the factor loadings for $\mathbf{x}_{i,t}$. Now, define the $m + 1$ vector $\mathbf{z}_{i,t}$ as

$$\mathbf{z}_{i,t} = \begin{pmatrix} y_{i,t} \\ \mathbf{x}_{i,t} \end{pmatrix}.$$

For this $\mathbf{z}_{i,t}$ we can obtain a factor model of the form

$$\mathbf{z}_{i,t} = \mathbf{C}_i\mathbf{f}_t + \mathbf{e}_{i,t}.$$

Obtain this factor model by using the models for $y_{i,t}$ and $\mathbf{x}_{i,t}$ and write $\mathbf{C}_i$ and $\mathbf{e}_{i,t}$ in terms the model components of the models for $y_{i,t}$ and $\mathbf{x}_{i,t}$.

Question 3: Asymptotic Derivations (25 points out of 100 points)

(a) (15 points) **Costs of ignoring constants and trends in the cointegrating regression:**

Suppose that we have the following data generating processes (DGP) for $\{y_t\}$ and $\{x_t\}$

$$y_t = \delta + \beta x_t + u_{1,t},$$
$$x_t = \mu + x_{t-1} + u_{2,t},$$

for $t = 1, \ldots, T$. We assume:

- $u_{1,t} = \psi_1(L)\epsilon_{1,t} = \sum_{j=0}^{\infty} \psi_{1,j}\epsilon_{1,t-j}$ where $\sum_{j=0}^{\infty} j \cdot |\psi_{1,j}| < \infty$ and $\{\epsilon_{1,t}\}$ is an $i.i.d$ sequence with mean zero and variance $\sigma^2_{\epsilon,1}$, and finite fourth moment, σ_1^2 denotes the long run variance of $\{u_{1,t}\}$ and $\sigma^2_{u,1}$ denotes the contemporaneous variance of $\{u_{1,t}\}$;
- $u_{2,t} = \psi_2(L)\epsilon_{2,t} = \sum_{j=0}^{\infty} \psi_{2,j}\epsilon_{2,t-j}$ where $\sum_{j=0}^{\infty} j \cdot |\psi_{2,j}| < \infty$ and $\{\epsilon_{2,t}\}$ is an $i.i.d$ sequence with mean zero and variance $\sigma^2_{\epsilon,2}$, and finite fourth moment, σ_2^2 denotes the long run variance of $\{u_{2,t}\}$ and $\sigma^2_{u,2}$ denotes the contemporaneous variance of $\{u_{2,t}\}$;
- $y_0 = x_0 = 0$.

A researcher wants to estimate β and for this purpose considers the following regression model

$$y_t = \beta x_t + error.$$

The OLS estimator is given by

$$\widehat{\beta} = \left(\sum_{t=1}^{T} x_t^2\right)^{-1} \sum_{t=1}^{T} x_t y_t.$$

The error of the estimator can be obtained by plugging in the model for y_t into this estimator. This will result in

$$\widehat{\beta} - \beta = \left(\sum_{t=1}^{T} x_t^2\right)^{-1} \sum_{t=1}^{T} x_t(\delta + u_{1,t}).$$

(i) Derive and describe the order of probability and the asymptotic distribution of $\widehat{\beta} - \beta$. Write down any necessary additional assumptions (if any).

(ii) Comment on your results. Is there a problem with this estimator?

(b) (10 points) **Costs of ignoring the time trend in the unit root AR(1) regression:**

Suppose that we have the following data generating process (DGP) for $\{y_t\}$

$$y_t = \delta + y_{t-1} + u_t$$

for $t = 1, \ldots, T$. We assume:

- $u_t = \psi(L)\epsilon_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$ where $\sum_{j=0}^{\infty} j \cdot |\psi_j| < \infty$ and $\{\epsilon_t\}$ is an $i.i.d$ sequence with mean zero and variance σ_ϵ^2, and finite fourth moment. σ_y^2 denotes the long run variance of $\{u_t\}$ and σ_u^2 denotes the contemporaneous variance of $\{u_t\}$;
- $y_0 = 0$;

We consider the estimation of the regression model

$$y_t = \lambda y_{t-1} + error,$$

using a sample of T observations.

Consider the least squares estimator

$$\widehat{\lambda} = \frac{\sum_{t=1}^{T} y_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2}.$$

The error of the estimator can be obtained by plugging in the model for y_t into this estimator. This will result in

$$\widehat{\lambda} - 1 = \left(\sum_{t=1}^{T} y_{t-1}^2 \right)^{-1} \sum_{t=1}^{T} y_{t-1}(\delta + u_{y,t}).$$

(i) Derive and discuss the orders of probability and limiting distributions of the numerator and the denominator of $(\widehat{\lambda} - 1)$. Write down any necessary additional assumptions (if any).

(ii) Comment on your results. Is there a problem with this estimator?

Question 4: Empirical Application (25 points out of 100 points)

(a) (15 points) Bruce Wayne is an econometrics student at Gotham City University. He wants to analyze the price elasticity of Batmobiles. He collects annual data on the following variables for the period 1921 - 2021:

- log of prices of Batmobiles ($price_t$),
- log of annual sales quantity of Batmobiles ($quant_t$)
- log of interest rates (int_t)
- log of annual inflation in Gotham (inf_t)

He knows from previous empirical research that all these variables contain unit roots. He wants to estimate long run relations between the variables. He starts his analysis by considering the trace test of Johansen. He produces the following output.

Null hypothesis	t-stat	5% critical values
$r = 0$	80.21	69.81
$r = 1$	64.44	47.85
$r = 2$	50.46	29.79
$r = 3$	10.27	15.49

Here r is the cointegration rank. Then, he considers a single equation regression to estimate the long run relation between these variables, he estimates by OLS the following regression

$$price_t = \beta_1 quant_t + \beta_2 int_t + \beta_3 inf_t + error.$$

(i) What conclusions can you draw from the results of the trace test.

(ii) Is there are problem with Bruce's approach of estimating a single static least squares regression? Explain in detail.

(iii) You know that he wants to estimate long run relations between the variables. What are your suggestions to Bruce? Explain in detail.

(b) (10 points) Harvey Dent is another econometrics student at Gotham City University. He is analyzing a panel data set of crime rates of 200 towns over 5 years. Let $c_{i,t}$ denote the number of crimes committed in town i. Previous research shows that crime rates data is dynamic in nature and it is highly suspected that an individual specific effect is present in the error term of the model that represents the heterogeneity across towns. Harvey realizes these and assumes the following model, where he denotes the variable by $c_{i,t}$:

$$c_{i,t} = \lambda c_{i,t-1} + u_{i,t},$$

where $u_{i,t} = \mu_i + \varepsilon_{i,t}$. He starts looking for an advice on how to analyze this panel data set.

(i) One of the ways to estimate this model is to use a fixed effects (within transformation) approach. However, it is known to be problematic. Explain why the fixed effects approach is problematic.

(ii) Propose and discuss briefly an alternative estimator that is more proper/optimal than the fixed effects estimator.