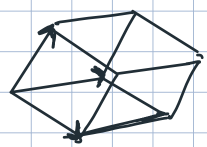
Answer Midterm 1, March 29, 2023

1.



$$Vol = \left| \det \begin{pmatrix} 1 & 1 & -3 \\ -1 & 3 & -1 \\ 3 & 5 & 7 \end{pmatrix} \right| =$$

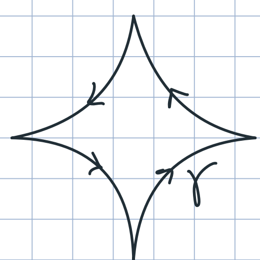
2

5

$$\left| \det \begin{pmatrix} 1 & 1 & -3 \\ 0 & 4 & -4 \\ 0 & 2 & 16 \end{pmatrix} \right| = \left| 1 \cdot \det \begin{pmatrix} 4 & -4 \\ 2 & 16 \end{pmatrix} \right| = 72$$

3

2.



(picture not necessary)

$$x = 2\cos^3\theta \qquad \theta \in [0, 2\pi]$$
$$y = 2\sin^3\theta$$

$$\ell(\gamma) = \int_0^{2\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta$$ 2

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -6\cos^2\theta \sin\theta \\ 6\sin^2\theta \cos\theta \end{pmatrix}$$ 2

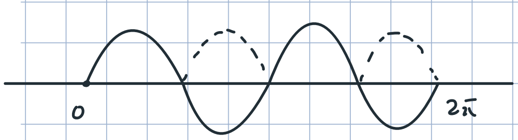
10

$$\Longrightarrow \ell(\gamma) = 6 \int_0^{2\pi} \sqrt{\cos^4\theta \sin^2\theta + \sin^4\theta \cos^2\theta}\, d\theta$$ 2

$$= 6 \int_0^{2\pi} |\cos\theta \sin\theta| \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta = 3 \int_0^{2\pi} |\sin 2\theta|\, d\theta$$

4

$$= 12 \int_0^{\pi/2} \sin 2\theta\, d\theta = -6\cos 2\theta \Big|_0^{\pi/2} = 12$$



3. $f(x,y) = (x+y)^2 \ln(x+y)$ about $(0,1)$.

$f(0,1) = 1^2 \ln 1 = 0$ 1

$f_x = 2(x+y)\ln(x+y) + (x+y)^2 \frac{1}{x+y} = 2(x+y)\ln(x+y) + x+y$

$f_y = 2(x+y)\ln(x+y) + x+y$ 2

$f_x(0,1) = f_y(0,1) = 1$

$f_{xx} = 2\ln(x+y) + 2 + 1 = 2\ln(x+y) + 3$

$f_{yy} = 2\ln(x+y) + 3$ 3

10 $f_{xy} = 2\ln(x+y) + 3$

$f_{xx}(0,1) = f_{yy}(0,1) = f_{xy}(0,1) = 3$

Taylor polynomial up to order 2

$P_f(x,y) = 0 + 1\cdot x + 1\cdot(y-1) + \frac{1}{2}3\cdot x^2 + 3\cdot x(y-1) + \frac{3}{2}(y-1)^2$

$= x + y - 1 + \frac{3}{2}x^2 + 3x(y-1) + \frac{3}{2}(y-1)^2$ 4

4.

$x^2 + y^2 s + y t^2 = 13$

$y^2 + x^2 s + x t^2 = 9$ $\quad (x,y,s,t) = (1,2,1,-2)$

Verify: $1^2 + 4\cdot 1 + 2\cdot 4 = 13$ ✓

$4 + 1\cdot 1 + 1\cdot 4 = 9$ ✓ 2

Seek: $s(x,y)$ and $t(x,y)$ near $(1,2,1,-2)$, i.e.

$s(1,2) = 1$, $t(1,2) = -2$ 1

diff. w.r.t. x:

15 $2x + y^2 s_x + 2yt t_x = 0$ 2

$2xs + x^2 s_x + t^2 + 2xt t_x = 0$

I: $\begin{pmatrix} y^2 & 2yt \\ x^2 & 2xt \end{pmatrix}\begin{pmatrix} s_x \\ t_x \end{pmatrix} = \begin{pmatrix} -2x \\ -2xs - t^2 \end{pmatrix}$ 1

w.r.t. y:

$2ys + y^2 s_y + t^2 + 2yt t_y = 0$ 2

$2y + x^2 s_y + 2xt t_y = 0$

$$\text{II}: \quad \begin{pmatrix} y^2 & 2yt \\ x^2 & 2xt \end{pmatrix} \begin{pmatrix} s_y \\ t_y \end{pmatrix} = \begin{pmatrix} -2ys - t^2 \\ -2y \end{pmatrix} \qquad 1$$

at $(1,2,1,-2)$ we have $\begin{pmatrix} 4 & 4\cdot(-2) \\ 1 & 2\cdot(-2) \end{pmatrix} = \begin{pmatrix} 4 & -8 \\ 1 & -4 \end{pmatrix}$

$\det\begin{pmatrix} 4 & -8 \\ 1 & -4 \end{pmatrix} = -16 + 8 = -8 \neq 0$ invertible $\quad 2$

and thus one can locally solve for (s_x, s_y) and (s_y, t_y).
By the Implicite Function Theorem $s(x,y)$ and $t(x,y)$ exist locally near $(x,y) = (1,2)$. $\quad 2$

To evaluate $s_y(1,2)$ we solve the linear system II.

$$\begin{pmatrix} 4 & -8 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} s_y \\ t_y \end{pmatrix} = \begin{pmatrix} -2\cdot 2\cdot 1 - 4 \\ -4 \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 0 & 8 & 8 \\ 1 & -4 & -4 \end{array}\right) \Longrightarrow \begin{array}{l} 8t_y = 8 \Rightarrow t_y = 1 \quad 2 \\ s_y - 4t_y = -4 \qquad s_y(1,2) = 0 \end{array}$$

Rmk:

$$\frac{\partial t}{\partial y} = -\frac{\frac{\partial(F_1,F_2)}{\partial(s,y)}}{\frac{\partial(F_1,F_2)}{\partial(s,t)}} = -\frac{\det\begin{pmatrix} \partial_s F_1 & \partial_y F_1 \\ \partial_s F_2 & \partial_y F_2 \end{pmatrix}}{\det\begin{pmatrix} \partial_s F_1 & \partial_t F_1 \\ \partial_s F_2 & \partial_t F_2 \end{pmatrix}} \Bigg|_{(1,2,1,-2)}$$

$$\begin{pmatrix} y^2 & 2ys+t^2 \\ x^2 & 2y \end{pmatrix} \rightsquigarrow \begin{pmatrix} 4 & 8 \\ 1 & 4 \end{pmatrix} \quad \det\begin{pmatrix} 4 & 4 \\ 1 & 4 \end{pmatrix} = 8$$

$\rightsquigarrow t_y = 1$

$$\begin{pmatrix} y^2 & 2yt \\ x^2 & 2xt \end{pmatrix} \rightsquigarrow \begin{pmatrix} 4 & -8 \\ 1 & -4 \end{pmatrix} \quad \det\begin{pmatrix} 4 & -8 \\ 1 & -4 \end{pmatrix} = -8$$

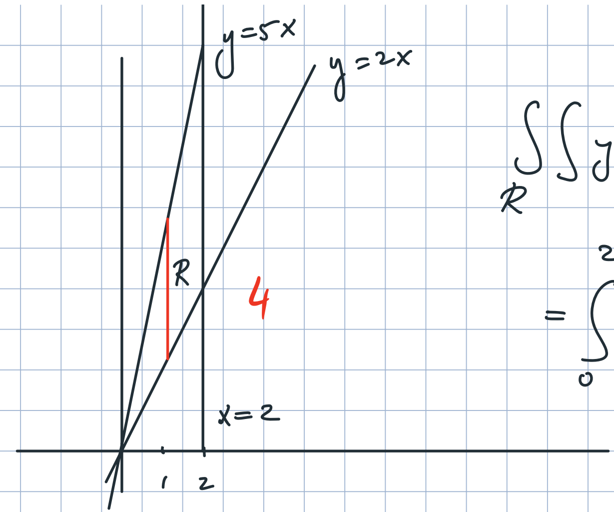
$$\frac{\partial s}{\partial y} = -\frac{\frac{\partial(F_1,F_2)}{\partial(y,t)}}{\frac{\partial(F_1,F_2)}{\partial(s,t)}} \qquad \frac{\partial(F_1,F_2)}{\partial(y,t)} = \det\begin{pmatrix} 2ys+t^2 & 2yt \\ 2y & 2xt \end{pmatrix}$$

$$\rightsquigarrow \det\begin{pmatrix} 8 & -8 \\ 4 & -4 \end{pmatrix} = 0$$

$\rightsquigarrow s_y = 0$

5.

(10)



4

$$\iint_R y^2\,dx\,dy = \int_{x=0}^{x=2}\int_{y=2x}^{y=5x} y^2\,dy\,dx$$ 2

$$= \int_0^2 \frac{1}{3}y^3\Big|_{2x}^{5x}\,dx = \int_0^2 \frac{1}{3}(125-8)x^3\,dx$$

$$= \frac{117}{3}\int_0^3 x^3\,dx$$

$$= \frac{117}{12}x^4\Big|_0^2 = \frac{117}{12}\cdot 16 = \frac{117}{3}\cdot 4 = 156$$ 4

6. Lagrange multiplier method.

$$L(x,y,\lambda) = x^2+y^2+\lambda(5x^2+6xy+5y^2-8)$$

$$\partial_x L = 2x + 10\lambda x + 6\lambda y = 0$$

$$\partial_y L = 2y + 6\lambda x + 10\lambda y = 0$$ 2

$$\partial_\lambda L = 5x^2+6xy+5y^2-8 = 0$$

$y\,\partial_x L - x\,\partial_y L = 0 \iff$

$6\lambda y^2 - 6\lambda x^2 = 0 \iff \lambda(y^2-x^2) = 0$ 4

(15) $\Rightarrow \lambda = 0$ or $y^2 = x^2$

$\underline{\lambda=0}$: $\Rightarrow x=0$ and $y=0$ contradicts $\partial_\lambda L = 0$ ↯ 2

$\underline{y^2=x^2}$: $x=y$ or $x=-y$

in $\partial_\lambda L = 0$ $\quad 5x^2+6x^2+5x^2 = 8$

$16x^2 = 8$, $x^2 = \frac{1}{2}$ $\quad x = \pm\frac{1}{2}\sqrt{2}$

$(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$ and $(-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$ 2

or $\quad 5x^2 - 6x^2 + 5x^2 = 8$

$4x^2 = 8 \Rightarrow x^2 = 2$, $\quad x = \pm\sqrt{2}$

$(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$ 2

$\left(\pm\frac{1}{2}\sqrt{2}\right)^2+\left(\pm\frac{1}{2}\sqrt{2}\right)^2=\frac{1}{2}+\frac{1}{2}=1$ min at $\left(\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2}\right)$ and $\left(\frac{1}{2}\sqrt{2},-\frac{1}{2}\sqrt{2}\right)$

3

$(\pm\sqrt{2})^2+(\mp\sqrt{2})^2=2+2=4$ max at $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$.

Rmk:

$$5x^2+6xy+5y^2=8$$

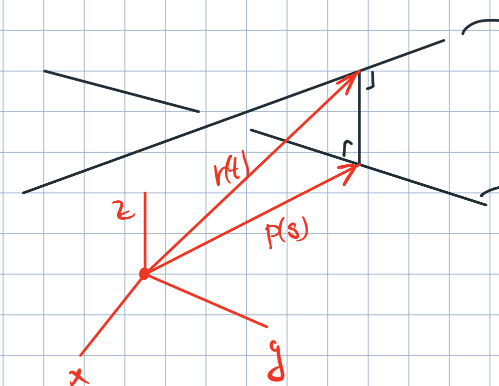
$$5\left(x^2+\frac{6}{5}xy+y^2\right)=8 \qquad \left(x+\frac{3}{5}y\right)^2=x^2+\frac{6}{5}xy+\frac{9}{25}y^2$$

$$5\left(\left(x+\frac{3}{5}y\right)^2-\frac{9}{25}y^2+y^2\right)=8$$

$$5\left(x+\frac{3}{5}y\right)^2+5\cdot\frac{16}{25}y^2=8$$

$$5\left(x+\frac{3}{5}y\right)^2+\frac{16}{5}y^2=8$$

7.



$x=-2z+4$
$y-2=-2z+4$

$$r(t)=\begin{pmatrix}4\\6\\0\end{pmatrix}+t\begin{pmatrix}-2\\-2\\1\end{pmatrix}$$

3

$x-3=-2y+6$
$z-1=-2y+6$

$$p(s)=\begin{pmatrix}9\\0\\7\end{pmatrix}+s\begin{pmatrix}-2\\1\\-2\end{pmatrix}$$

3

15

$$r(t)-p(s)=\begin{pmatrix}-5+2s-2t\\6-s-2t\\-7+2s+t\end{pmatrix}\perp\begin{pmatrix}-2\\-2\\1\end{pmatrix} \text{ and } \perp\begin{pmatrix}-2\\1\\-2\end{pmatrix}$$

via inner product $\Longrightarrow$

3 3

$-9+9t=0 \Rightarrow t=1$ $\qquad 30-9s=0$

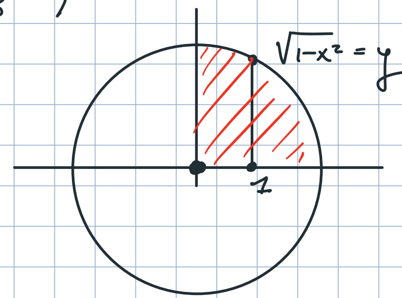
$\Longrightarrow s=\frac{30}{9}=\frac{10}{3}$

$$r(1)-p\left(\frac{10}{3}\right)=\begin{pmatrix}-5+\frac{20}{3}-2\\6-\frac{10}{3}-2\\-7+\frac{20}{3}+1\end{pmatrix}=\begin{pmatrix}-1/3\\2/3\\2/3\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-1\\2\\2\end{pmatrix}$$

3

$$d = \left\| r(1) - p\left(\tfrac{10}{3}\right) \right\| = \frac{1}{3}\sqrt{1+4+4} = \frac{1}{3}\sqrt{9} = 1.$$

8. $$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)^{5/2}\,dx\,dy$$



4

In polar coordinates:

$x = r\cos\theta$
$y = r\sin\theta$ $\rightsquigarrow$ $r \in [0,1]$, $\theta \in [0, \frac{\pi}{2}]$

10

$dx\,dy = r\,dr\,d\theta$ 4

$$\Longrightarrow \int_0^{\pi/2}\int_0^1 (r^2)^{5/2} \cdot r\,dr\,d\theta = \int_0^{\pi/2} d\theta \int_0^1 r^6\,dr$$

$$= \pi/2 \cdot 1/7 = \frac{\pi}{14}$$ 2

9. $$\int_{y=0}^{y=1}\int_{z=0}^{z=y}\int_{x=0}^{x=z} 30xy\,dx\,dz\,dy = \int_0^1\int_0^y 15x^2y \Big|_0^z dz\,dy$$

4 2

$$= \int_0^1\int_0^y 15z^2y\,dz\,dy = \int_0^1 5z^3y\Big|_0^y dy = \int_0^1 5y^4\,dy$$

10 2

$$= y^5\Big|_0^1 = 1.$$

2

100